Reality and its representations: a mathematical model

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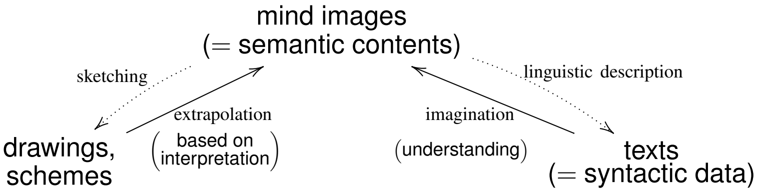
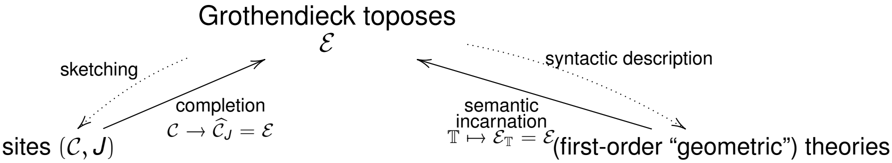
The double expression of semantic contents
and its modelling by topos theory:



Proposed mathematical model:



$=$ "small" category $\mathcal{C}$

$+ J =$ topology

$= $ extrapolation principle on $\mathcal{C}$

$\mathbb{T}$

$=$ vocabulary (words)

$+$ axioms (grammar rules)

Why can we propose Grothendieck toposes for modelling elements of reality?

- For us human beings:
  - Any aspects or elements of reality
    can be described or at least talked about
    by appropriate forms of human language.
  - On the other hand, these linguistic descriptions are not unique.
    Reality is independent of its multiple descriptions.
- In topos theory:
  - Any topos $\mathcal{E}$ can be presented as
    a geometric incarnation of the semantic contents
    of some formalized language $\mathbb{T}$
    (technically, a "first-order geometric theory")
    in the sense that there is an identification

$$\{\text{points of the topos } \mathcal{E}\} \leftrightarrow \{\text{models of the theory } \mathbb{T}\}.$$

  - Such a linguistic description of a topos $\mathcal{E}$ is not unique.
    Any topos $\mathcal{E}$ incarnates the semantics of infinitely many theories $\mathbb{T}$.
  - This correspondence is complete in the sense that
    the semantics of any such theory $\mathbb{T}$ is incarnated by a topos $\mathcal{E}_{\mathbb{T}}$.

Geometric sketching of toposes:

- Start with an element of reality or semantic content which is supposed to be mathematically incarnated by
  an unknown topos $\mathcal{E}$.
- Technically, a topos is a special type of "category" = "mathematical country" consisting in
  $$\begin{cases} - & \text{cities } A, B, C, \cdots \\ - & \text{itineraries } A \to B \text{ between cities,} \\ - & \text{a law for composing itineraries } A \to B \to C. \end{cases}$$
- As a category, a topos is "complete" in the sense that anything which can be mathematically extrapolated from elements of the topos exists in the topos.
- As it is complete, a topos $\mathcal{E}$ is "too big".
- It needs to be approximated by "small" categories
  $$\mathcal{C} \longrightarrow \mathcal{E}.$$
- A full topos $\mathcal{E}$ can be reconstructed from a small category
  $$\mathcal{C} \longrightarrow \mathcal{E} \quad \text{if } \mathcal{C} \text{ is "dense" in } \mathcal{E}.$$
- In that case, there is a unique "topology" (= extrapolation principle)
  $$J \text{ on } \mathcal{C} \ \text{ such that } \ \widehat{\mathcal{C}}_J \xrightarrow{\sim} \mathcal{E}.$$

Linguistic description of toposes:

- Starting with an unknown topos $\mathcal{E}$,
  suppose we have identified enough elements of $\mathcal{E}$
  to define a sketching by a small category $\mathcal{C} \longrightarrow \mathcal{E}$.
- Suppose this sketching is "dense" so that
  $\widehat{\mathcal{C}}_J \xrightarrow{\sim} \mathcal{E}$ for some topology J.
- For a linguistic description of $\mathcal{E}$, we need a (first-order geometric) theory $\mathbb{T}$
  which is well-adapted to talk about $\mathcal{E}$ and $\mathcal{C}$
  in the sense that there exists a natural
  "naming functor"
  $\mathcal{C} \longrightarrow \mathcal{C}_{\mathbb{T}}$ = "syntactic category" of $\mathbb{T}$ consisting in
  $\begin{cases} \text{cities} = \text{formulas in the vocabulary of } \mathbb{T}, \\ \text{itineraries} = \mathbb{T}\text{-provable implications} \end{cases}$

  inducing a topos morphism $\widehat{\mathcal{C}}_J \longrightarrow \widehat{(\mathcal{C}_{\mathbb{T}})}_{J_{\mathbb{T}}} = \mathcal{E}_{\mathbb{T}}$.
- If this morphism is an embedding $\widehat{\mathcal{C}}_J \hookrightarrow \mathcal{E}_{\mathbb{T}}$,
  there is a "quotient theory" $\mathbb{T}'$ of $\mathbb{T}$
  (with the same vocabulary and more axioms) such that
  $\widehat{\mathcal{C}}_J \xrightarrow{\sim} \mathcal{E}_{\mathbb{T}'}$, so that $\mathbb{T}'$ describes $\widehat{\mathcal{C}}_J \cong \mathcal{E}$.

Partial sketching of toposes:

- Starting from an unknown topos $\mathcal{E}$
(incarnating some element of reality or semantic content),
we would want to draw a "dense" sketch

$$\mathcal{C} \longrightarrow \mathcal{E}$$

by a category $\mathcal{C}$ which is
finite (or at least can be described with finitely many words).
- This is not possible in general.
- This means that we have to accept partial sketchings

$$\mathcal{C} \longrightarrow \mathcal{E}$$

which are not dense:
there is no equivalence $\widehat{\mathcal{C}}_J \xrightarrow{\sim} \mathcal{E}$.
- Theoretically, such a $\mathcal{C} \to \mathcal{E}$ defines a canonical topology J on $\mathcal{C}$
inducing a topos morphism $\widehat{\mathcal{C}}_J \to \mathcal{E}$.
But it cannot be constructed on $\mathcal{C}$ as $\mathcal{E}$ is not known.
- This means that the interpretation of $\mathcal{C}$
(incarnated in a topology $J =$ extrapolation principle)
cannot come from $\mathcal{E}$.

Joint descriptions of toposes of some type:

- Suppose we start from a family of toposes

$$\mathcal{E}_i\,, \qquad i \in I\,,$$

  which incarnate elements of reality of the same type.
  $\rightarrow$ For instance, all $\mathcal{E}_i$'s could be real images
  which we want to sketch and describe.
- As all $\mathcal{E}_i$'s incarnate elements of reality
  of the same type,
  it is natural to think that there should exist
  a joint description theory $\mathbb{T}$
  for all $\mathcal{E}_i$'s.
- This means that each $\mathcal{E}_i$ could be
  partially (but quite faithfully) sketched
  by a finite category

$$\mathcal{C}_i \longrightarrow \mathcal{E}_i$$

  endowed with a naming functor

$$N_i : \mathcal{C}_i \longrightarrow \mathcal{C}_{\mathbb{T}}\,.$$

Interpretation through language:

- Suppose that there is a general formalized language $\mathbb{T}$ for describing elements of reality of some type, incarnated in a family of toposes $\mathcal{E}_i$, $i \in I$.
- This means that there are (quite faithful) sketches

$$\mathcal{C}_i \longrightarrow \mathcal{E}_i$$

by finite categories $\mathcal{C}_i$ endowed with naming functors

$$N_i : \mathcal{C}_i \longrightarrow \mathcal{C}_{\mathbb{T}} \,.$$

- For each i, the syntactic topology $J_{\mathbb{T}}$ of $\mathcal{C}_{\mathbb{T}}$ (characterized by $\widehat{(\mathcal{C}_{\mathbb{T}})}_{J_{\mathbb{T}}} = \mathcal{E}_{\mathbb{T}}$) induces a canonical topology J_i on $\mathcal{C}_i$ defining a cartesian square of toposes:

$$\begin{array}{ccc} \widehat{(\mathcal{C}_i)}_{J_i} & \longrightarrow & \widehat{(\mathcal{C}_{\mathbb{T}})}_{J_{\mathbb{T}}} \\ \big\downarrow & & \big\downarrow \\ \widehat{\mathcal{C}}_i & \longrightarrow & \widehat{\mathcal{C}}_{\mathbb{T}} \end{array}$$

- This means that the interpretation J_i on $\mathcal{C}_i$ would come from the general description theory $\mathbb{T}$.

General language and partial singular descriptions:

- Suppose that each topos $\mathcal{E}_i$ in the family can be sketched by a finite category
  $$\mathcal{C}_i \longrightarrow \mathcal{E}_i$$
  endowed with a naming functor
  $$N_i : \mathcal{C}_i \longrightarrow \mathcal{C}_{\mathbb{T}}$$
  so that the topology $J_{\mathbb{T}}$ of $\mathcal{C}_{\mathbb{T}}$ induces an interpretation topology J_i on $\mathcal{C}_i$.
- Each induced morphism of toposes
  $$\widehat{(\mathcal{C}_i)}_{J_i} \longrightarrow \widehat{(\mathcal{C}_{\mathbb{T}})}_{J_{\mathbb{T}}} = \mathcal{E}_{\mathbb{T}}$$
  factorizes canonically as
  $$(\widehat{\mathcal{C}_i})_{J_i} \xrightarrow{\text{surjection}} \mathcal{E}_{\mathbb{T}_i} \xhookrightarrow{\text{embedding}} \mathcal{E}_{\mathbb{T}}$$
  for a unique subtopos $\mathcal{E}_{\mathbb{T}_i}$ of $\mathcal{E}_{\mathbb{T}}$
  which incarnates the semantic content
  of a unique quotient theory $\mathbb{T}_i$ of $\mathbb{T}$:
  the theory $\mathbb{T}_i$ has the same vocabulary as $\mathbb{T}$,
  but has more axioms.
- The extra axioms of $\mathbb{T}_i$ make up a singular partial description of $\mathcal{E}_i$.

Defining a description language:

- Start with a family of elements of reality considered of the same type.
  $\rightarrow$ For example: images.
- This similarity should be expressed in the form of a joint description theory $\mathbb{T}$.
- If each "element of reality" in the family is considered to be incarnated by an unknown topos $\mathcal{E}_i$, we need the vocabulary and the axioms of $\mathbb{T}$ to be rich enough so that:
  - Each $\mathcal{E}_i$ can be sketched by a finite category

    $$\mathcal{C}_i \longrightarrow \mathcal{E}_i$$

    endowed with a naming functor

    $$N_i : \mathcal{C}_i \longrightarrow \mathcal{C}_{\mathbb{T}} \,.$$

  - The topology J_i on $\mathcal{C}_i$ induced by the topology $J_{\mathbb{T}}$ of $\mathcal{C}_{\mathbb{T}}$ should be refined enough to define a topos morphism

    $$\widehat{(\mathcal{C}_i)}_{J_i} \longrightarrow \mathcal{E}_i \,.$$

Starting from a vocabulary without axioms:

- Starting from a family of elements of reality
  supposed to be incarnated by some unknown toposes $\mathcal{E}_i$, $i \in I$,
  one may first define a vocabulary Σ
  rich enough so that:

$$\begin{cases} \text{Each } \mathcal{E}_i \text{ can be sketched by a finite category} \\ \qquad\qquad \mathcal{C}_i \longrightarrow \mathcal{E}_i \\ \text{endowed with a naming functor} \\ \qquad\qquad N_i : \mathcal{C}_i \longrightarrow \mathcal{C}_\Sigma \, . \end{cases}$$

- Then, one may look for a topology J on $\mathcal{C}_\Sigma$
  such that, for any $i \in I$, the induced topology J_i on $\mathcal{C}_i$
  defines a topos morphism

$$\widehat{(\mathcal{C}_i)}_{J_i} \longrightarrow \mathcal{E}_i \, .$$

  This means that any point of $\widehat{(\mathcal{C}_i)}_{J_i}$ should make sense as a point of $\mathcal{E}_i$.
- Such a topology J on $\mathcal{C}_\Sigma$
  corresponds to a quotient theory $\mathbb{T}$ of Σ
  (defined by the same vocabulary completed with axioms)
  which is a description theory for the $\mathcal{E}_i$'s.