Some sketches for a topos-theoretic AI

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The problem of representing elements of reality :

- Question:
  How can elements of reality
  be represented in computers?
- Conditions to verify:
  → These representations have to allow storage.
  → They also have to allow processing,
  especially processes of information extraction.
- First consequences:
  → Representations of elements of reality in computers
  have to consist in finite sequences of symbols,
  or equivalent other types of data.
  → The different symbols that may appear can be called words.
  (Rk: Any number is a name, but names are not necessarily numbers.)
  → Grammar rules (= "axioms"),
  expressing identities or equivalences between different expressions,
  are needed to allow processing, i.e. computing.
- The need for formal languages:
  → The list of symbols (names) and axioms (gammar rules)
  has to be fully explicit.

The problem of representing families of elements of reality:

- Question:
  How to represent elements of reality
  in a way which expresses the fact that they belong to a common family?
  (For example: how to represent images?)
- Conditions to verify:
  → We need to express the fact that
  some series of elements of reality belong to the same family.
  → At the same time, we need
  to represent different elements of reality
  by different representations.
- What our experience with natural languages teaches us:
  → A natural way to express the fact that
  a series of elements of reality belong to the same family
  is to describe them by using the same vocabulary.
  → In other words, they should be represented
  by a joint description language,
  consisting in a joint vocabulary and joint grammar rules.
  → Each particular element of reality should be distinguished
  by additional specific grammar rules (ex: setting coordinates).

A general notion of formalized language:

Definition. – *A first-order "geometric" theory $\mathbb{T}$ is a datum of*

(1) *a vocabulary Σ consisting in*

- *a family of "sorts" (= object names) E_i, $i \in I$,*
  such as for instance "group G", "ring A", "module M", $\cdots$
- *a family of "function symbols" (= map names) $E_1 \cdots E_n \xrightarrow{f} E$,*

  such as for instance $GG \xrightarrow{\cdot} G$, $G \xrightarrow{(\bullet)^{-1}} G$,

  or $AA \xrightarrow{+} A$, $AA \xrightarrow{\cdot} A$, $A \xrightarrow{-(\bullet)} A, \cdots$
- *a family of "relation symbols" (= relation names) $R \rightarrowtail E_1 \cdots E_n$,*
  such as for instance $\leq \rightarrowtail EE$, $\sim \rightarrowtail EE, \cdots$

(2) *a family of "axioms" which consist in implications*

$$\varphi(\vec{x}) \vdash \psi(\vec{x}) \quad \textit{where}$$

- $\vec{x} = (x_1^{E_1} \cdots x_n^{E_n})$ *is a finite family of "variables" $x_i^{E_i}$*
  associated with sorts E_i,
- φ, ψ *are "formulas" in the variables $x_1^{E_1} \cdots x_n^{E_n}$*
  which are constructed from function or relation symbols
  and can be interpreted in terms of
  "images of maps", "arbitrary unions of subobjects" and "finite intersections".

The notion of model:

The usual relationship between

$$\begin{cases} - & \text{natural languages (vocabulary + grammar rules),} \\ - & \text{elements of reality to which natural languages apply,} \end{cases}$$

inspires the mathematical relationship between

$$\begin{cases} - & \text{formal languages } \mathbb{T} \text{ (vocabulary } \Sigma + \text{ list of "axioms"),} \\ - & \text{"models" of } \mathbb{T}. \end{cases}$$

Definition. – *A set-valued model M of a (first-order geometric) theory $\mathbb{T}$ is a triple map*

$$\begin{cases} \bullet & \text{any sort } E \mapsto \text{set } ME, \\ \bullet & \text{any function symbol } (E_1 \cdots E_n \xrightarrow{f} E) \mapsto \text{map } ME_1 \times \cdots \times ME_n \xrightarrow{Mf} ME, \\ \bullet & \text{any relation symbol } (R \rightarrowtail E_1 \cdots E_n) \mapsto \text{subset } MR \hookrightarrow ME_1 \times \cdots \times ME_n, \end{cases}$$

such that, for any axiom of $\mathbb{T}$

$$\varphi(x_1^{E_1} \cdots x_n^{E_n}) \vdash \psi(x_1^{E_1} \cdots x_n^{E_n}) \quad ,$$

the interpretations of the formulas φ, ψ as subsets

$$\begin{array}{rcl} M\varphi & \hookrightarrow & ME_1 \times \cdots \times ME_n \,, \\ M\psi & \hookrightarrow & ME_1 \times \cdots \times ME_n \end{array}$$

verify

$$M\varphi \subseteq M\psi \,.$$

Geometric models:

Proposition. – *The notion of model M of a "geometric theory" $\mathbb{T}$ as a map*

$$\begin{cases} \bullet \quad \textit{sort } E \mapsto \textit{object } ME, \\ \bullet \quad \textit{function symbol } (E_1 \cdots E_n \xrightarrow{f} E) \mapsto \textit{morphism } (ME_1 \times \cdots \times ME_n \xrightarrow{Mf} ME), \\ \bullet \quad \textit{relation symbol } (R \rightarrowtail E_1 \cdots E_n) \mapsto \textit{subobject } (MR \hookrightarrow ME_1 \times \cdots ME_n) \end{cases}$$

makes sense in any locally small category $\mathcal{C}$
which is "geometric" in the sense that

$$\begin{cases} \bullet \quad \textit{finite products } X_1 \times \cdots \times X_n \textit{ and fiber products } S' \times_S X \textit{ (for } \begin{array}{ccc} & & X \\ & & \downarrow \\ S' & \to & S \end{array}\textit{)} \\ \quad\ \textit{are well-defined in } \mathcal{C}, \\ \bullet \quad \textit{morphisms } X' \xrightarrow{f} X \textit{ have well-defined images } \mathrm{Im}(f) \hookrightarrow X, \\ \bullet \quad \textit{arbitrary unions and finite intersections of subobjects are well-defined,} \\ \bullet \quad \textit{fiber products functors } S' \times_S \bullet \textit{ respect images, unions and intersections.} \end{cases}$$

Remarks :

• A model M of $\mathbb{T}$ in a "geometric category" $\mathcal{C}$ is defined by the property that, for any axiom $\varphi(x_1^{E_1} \cdots x_n^{E_n}) \vdash \psi(x_1^{E_1} \cdots x_n^{E_n})$, the interpretations as subobjects $M\varphi, M\psi \hookrightarrow ME_1 \times \cdots \times ME_n$ verify the relation $M\varphi \subseteq M\psi$.

• Models of $\mathbb{T}$ in a geometric category $\mathcal{C}$ make up a locally small category $\mathbb{T}\text{-mod}(\mathcal{C})$.

Diagrammatic models:

Definition. – *For any (essentially) small category $\mathcal{C}$, the associated category of "presheaves" or $\mathcal{C}$-indexed diagrams of sets is*

$$\widehat{\mathcal{C}} = [\mathcal{C}^{\mathrm{op}}, \mathrm{Set}] = \{\text{category of functors } \mathcal{C}^{\mathrm{op}} \to \mathrm{Set}\}.$$

Most important reminder:

$\mathcal{C}$ and $\widehat{\mathcal{C}}$ are related by the fully faithful Yoneda functor

$$\begin{cases} \mathcal{C} & \hookrightarrow & \widehat{\mathcal{C}}\,, \\ X & \longmapsto & \mathrm{Hom}(\bullet, X)\,. \end{cases}$$

Proposition. –

(i) *Any such category of diagrams $\widehat{\mathcal{C}}$ on some $\mathcal{C}$ shares all constructive properties of the category Set. In particular, it is geometric.*

(ii) *For any geometric theory $\mathbb{T}$, the category of models*

$$\mathbb{T}\text{-mod}\,(\widehat{\mathcal{C}})$$

identifies with the category

$$[\mathcal{C}^{\mathrm{op}}, \mathbb{T}\text{-mod}\,(\mathrm{Set})]$$

of diagrams of set-valued models of $\mathbb{T}$

$$\mathcal{C}^{\mathrm{op}} \longrightarrow \mathbb{T}\text{-mod}\,(\mathrm{Set}).$$

Continuous family of models:

Definition. – *Let $\mathbb{T}$ be a geometric theory.*

(i) *The category of continuous family of $\mathbb{T}$-models parametrized by a topological space X is*

$$\mathbb{T}\text{-mod}\,(\mathcal{E}_X)$$

where $\mathcal{E}_X =$ category of set-valued "sheaves" on X.

(ii) *For any point $x \in X$, the functor of evaluation at x of these models*

$$x^* : \mathbb{T}\text{-mod}\,(\mathcal{E}_X) \longrightarrow \mathbb{T}\text{-mod}\,(\text{Set})$$

is induced by the fiber functor at x

$$x^* : \mathcal{E}_X \longrightarrow \text{Set}\,.$$

(iii) *More generally, for any continuous map $f : X' \to X$, the functor of change of parameters by $f : X' \to X$ is*

$$f^* : \mathbb{T}\text{-mod}\,(\mathcal{E}_X) \longrightarrow \mathbb{T}\text{-mod}\,(\mathcal{E}_{X'})$$

induced by $f^* : \mathcal{E}_X \longrightarrow \mathcal{E}_{X'}\,.$

Justification:

- If O_X = category of open subsets $U \hookrightarrow X$ and inclusions $U \subseteq U'$, $\mathcal{E}_X$ is defined as the full subcategory of $\widehat{O(X)} = \{\text{presheaves on } O(X)\}$ on "sheaves" = presheaves which verify a "glueing condition" w.r.t. coverings.
- The category of sheaves $\mathcal{E}_X$ shares all constructive properties of Set. In particular, it is geometric.
- Any continuous map $X' \xrightarrow{f} X$ defines a pair of adjoint functors

$$(\mathcal{E}_X \xrightarrow{f^*} \mathcal{E}_{X'}, \mathcal{E}_{X'} \xrightarrow{f_*} \mathcal{E}_X$$

such that

$$\begin{cases} - \quad f_* \text{ is composition with } f^{-1} : O_X \to O_{X'}, \\ - \quad f^* \text{ respects colimits (sums and quotients)} \\ \quad\ \ \text{and finite limits (finite products and fiber products).} \end{cases}$$

A joint generalization of categories of diagrams and categories of sheaves:

Definition. – *Let $\mathcal{C}$ be an essentially small category.*

(i) *A topology J is a notion of covering families*

$$(X_i \xrightarrow{x_i} X)_{i \in I} \text{ of objects } X \text{ of } \mathcal{C},$$

such that:

$$\begin{cases} (A) & \text{For any } X,\ X \xrightarrow{\mathrm{id}_X} X \text{ is a covering.} \\ (B) & \text{Any morphism } X' \to X \text{ transforms coverings of } X \\ & \text{into coverings of } X'. \\ (C) & \text{For any covering family } (X_i \xrightarrow{x_i} X)_{i \in I}, \\ & \text{its composites with families of coverings } (X_{i,k} \xrightarrow{x_{i,k}} X_i)_{k \in K_i} \\ & \text{make up a covering } (X_{i,k} \xrightarrow{x_i \circ x_{i,k}} X)_{i \in I, k \in K_i}. \end{cases}$$

(ii) *A J-sheaf on $\mathcal{C}$ is a presheaf $\mathcal{C}^{\mathrm{op}} \to \mathrm{Set}$ which verify a "glueing condition" w.r.t. J-coverings.*

(iii) *J-sheaves on $\mathcal{C}$ make up a full subcategory*

$$\widehat{\mathcal{C}}_J \hookrightarrow \widehat{\mathcal{C}} = \text{category of presheaves.}$$

(iv) *A topos is a category $\mathcal{E}$ which is equivalent to categories of sheaves $\widehat{\mathcal{C}}_J$.*

Topos-valued models:

Proposition. –
Any topos $\mathcal{E}$ has the same constructive properties as Set.
In particular, it is geometric.

Consequence:
Any geometric theory $\mathbb{T}$ defines categories of models
$\mathbb{T}\text{-mod}(\mathcal{E})$ indexed by toposes $\mathcal{E}$

Definition. –
A morphism of toposes $f : \mathcal{E}' \to \mathcal{E}$ is a pair of adjoint functors

$$(\mathcal{E} \xrightarrow{f^*} \mathcal{E}, \; \mathcal{E}' \xrightarrow{f_*} \mathcal{E})$$

such that f^* respects not only arbitrary colimits but also finite limits.

Remark: If $X, X' =$ topological spaces and X is “sober”,
morphisms of toposes $\mathcal{E}_{X'} \to \mathcal{E}_X$ correspond to continuous maps $X' \to X$.

Consequence: For any geometric theory $\mathbb{T}$, morphisms of toposes $f : \mathcal{E}' \to \mathcal{E}$
induce functors of change of parameters of $\mathbb{T}$-models

$$f^* : \mathbb{T}\text{-mod}(\mathcal{E}) \longrightarrow \mathbb{T}\text{-mod}(\mathcal{E}') \, .$$

The topological incarnation of the semantics of a formal language:

For a geometric theory $\mathbb{T}$,
its semantics consists in the network of categories of models

$\mathbb{T}\text{-mod}(\mathcal{E})$ parametrized by toposes $\mathcal{E}$

and related by functors of change of parameters
$f^* : \mathbb{T}\text{-mod}(\mathcal{E}) \to \mathbb{T}\text{-mod}(\mathcal{E}')$ defined by morphisms $f : \mathcal{E}' \to \mathcal{E}$.

Theorem. – *For a geometric theory $\mathbb{T}$, its semantics is incarnated in a topos*

$$\mathcal{E}_{\mathbb{T}} = \text{“classifying topos” of } \mathbb{T}$$

endowed with a $\mathbb{T}$-model in $\mathbb{T}\text{-mod}(\mathcal{E}_{\mathbb{T}})$

$$U_{\mathbb{T}} = \text{“universal model” of } \mathbb{T},$$

characterized by the property that, for any topos $\mathcal{E}$,
changes of parameters by morphisms of toposes $f : \mathcal{E} \to \mathcal{E}_{\mathbb{T}}$
define an equivalence of categories

$$\begin{cases} \text{Geom}(\mathcal{E}, \mathcal{E}_{\mathbb{T}}) & \xrightarrow{\sim} & \mathbb{T}\text{-mod}(\mathcal{E})\,, \\ (\mathcal{E} \xrightarrow{f} \mathcal{E}_{\mathbb{T}}) & \longmapsto & f^* U_{\mathbb{T}}\,. \end{cases}$$

Remark :
For any toposes $\mathcal{E}, \mathcal{E}'$, morphisms $\mathcal{E}' \xrightarrow{f} \mathcal{E}$ make up a category $\text{Geom}(\mathcal{E}', \mathcal{E})$
whose morphisms $f_1 \to f_2$ are by definition natural transformations

$$\rho : f_1^* \longrightarrow f_2^*\,.$$

The topological interpretation of syntax:

Let $\mathbb{T}$ be a geometric theory and Σ its vocabulary.
Let $\mathcal{E}_{\mathbb{T}}$ be its "classifying topos" and $U_{\mathbb{T}}$ be its "universal model".

Theorem. –

(i) Sorts E of Σ interpret as objects $U_{\mathbb{T}}E$ of $\mathcal{E}_{\mathbb{T}}$.

Symbols of functions $E_1 \cdots E_n \xrightarrow{f} E$ interpret as morphisms $U_{\mathbb{T}}E_1 \times \cdots \times U_{\mathbb{T}}E_n \xrightarrow{U_{\mathbb{T}}f} U_{\mathbb{T}}E$.

Symbols of relations $R \rightarrowtail E_1 \cdots E_n$ interpret as subobjects $U_{\mathbb{T}}R \hookrightarrow U_{\mathbb{T}}E_1 \times \cdots \times U_{\mathbb{T}}E_n$.

(ii) Formulas $\varphi(x_1^{E_1} \cdots x_n^{E_n})$ interpret as subobjects $U_{\mathbb{T}}\varphi \hookrightarrow U_{\mathbb{T}}E_1 \times \cdots \times U_{\mathbb{T}}E_n$.

Conversely, all subobjects of $U_{\mathbb{T}}E_1 \times \cdots \times U_{\mathbb{T}}E_n$ are interpretations of such formulas.

(iii) An implication between two formulas

$$\varphi(x_1^{E_1} \cdots x_n^{E_n}) \vdash \psi(x_1^{E_1} \cdots x_n^{E_n}) \text{ is } \mathbb{T}\text{-provable}$$

if and only if $U_{\mathbb{T}}\varphi \subseteq U_{\mathbb{T}}\psi$ as subobjects of $U_{\mathbb{T}}E_1 \times \cdots \times U_{\mathbb{T}}E_n$.

(iv) For any two formulas $\varphi(\vec{x})$ and $\psi(\vec{y})$, the graphs of the morphisms $U_{\mathbb{T}}\varphi \to U_{\mathbb{T}}\psi$ in $\mathcal{E}_{\mathbb{T}}$ are the interpretations $U_{\mathbb{T}}\theta$ of formulas

$\theta(\vec{x}, \vec{y})$ which are "provably functional" w.r.t. φ and ψ.

(v) A family of morphisms $\varphi(\vec{x}_i) \xrightarrow{\theta_i(\vec{x}_i, \vec{y})} \psi(\vec{y})$, $i \in I$, is a covering if and only if

$$\psi(\vec{y}) \vdash \bigvee_{i \in I} (\exists \vec{x}_i)\, \theta_i(\vec{x}_i, \vec{y}) \qquad \text{is } \mathbb{T}\text{-provable.}$$

(vi) This defines a site $(\mathcal{C}_{\mathbb{T}}, J_{\mathbb{T}})$ (called the "syntactic site" of $\mathbb{T}$) such that

$$\mathcal{E}_{\mathbb{T}} \cong \widehat{(\mathcal{C}_{\mathbb{T}})}_{J_{\mathbb{T}}}\,.$$

Points of toposes and their linguistic descriptions:

Definition. –

(i) *The category of points of a topos $\mathcal{E}$ is defined as*

$$\mathrm{pt}(\mathcal{E}) = \mathrm{Geom}(\mathrm{Set}, \mathcal{E}) = \{\textit{category of topos morphisms } \mathrm{Set} \to \mathcal{E}\}.$$

(ii) *More generally, the category of $\mathcal{E}'$-parametrized points of $\mathcal{E}$ is*

$$\mathrm{Geom}(\mathcal{E}', \mathcal{E}) = \{\textit{category of topos morphisms } \mathcal{E}' \to \mathcal{E}\}.$$

Theorem. –

Any presentation of a topos $\mathcal{E}$ by a site

$$\mathcal{E} = \widehat{\mathcal{C}}_J$$

defines a geometric theory $\mathbb{T}_{\mathcal{C},J}$ such that:

(1)
- *the sorts of $\mathbb{T}_{\mathcal{C},J}$ are the objects X of $\mathcal{C}$,*
- *the function symbols of $\mathbb{T}_{\mathcal{C},J}$ are the morphisms $X' \to X$ of $\mathcal{C}$,*
- *$\mathbb{T}_{\mathcal{C},J}$ has no relation symbol,*

(2) *for any topos $\mathcal{E}'$, the category of $\mathcal{E}'$-parametrized points*

$$\mathrm{Geom}(\mathcal{E}', \mathcal{E})$$

identifies with the category of models

$$\mathbb{T}_{\mathcal{C},J}(\mathcal{E}').$$

Subtoposes, topologies and quotient theories:

Definition. – *A subtopos of a topos $\mathcal{E}$ is a morphism of toposes*

$$\mathcal{E}' \xhookrightarrow{j} \mathcal{E}$$

whose push-forward component $j_* : \mathcal{E}' \to \mathcal{E}$ is fully faithful.

Theorem (SGA 4). – *If a topos $\mathcal{E}$ is presented by a site as $\mathcal{E} \cong \widehat{\mathcal{C}}_J$, there is a one-to-one correspondence between*

- *subtoposes $\mathcal{E}' \hookrightarrow \mathcal{E}$,*
- *topologies J' on $\mathcal{C}$ which are more refined than J.*

In particular, if $J' \supseteq J$, $\widehat{\mathcal{C}}_{J'} \hookrightarrow \widehat{\mathcal{C}}_J$ is a subtopos.

Theorem (Caramello's PhD thesis). –
If a topos $\mathcal{E}$ is presented by a geometric theory $\mathbb{T}$ as $\mathcal{E} \cong \mathcal{E}_{\mathbb{T}}$, there is a one-to-one correspondence between

- *subtoposes $\mathcal{E}' \hookrightarrow \mathcal{E}$,*
- *"quotient" theories $\mathbb{T}'$ of $\mathbb{T}$, up to syntactic equivalence.*

Explanation :

- A "quotient" theory is a theory in the same vocabulary with more axioms.
- Two theories in the same vocabulary Σ are "syntactically equivalent" if they yield the same collection of provable implications $\varphi(\vec{x}) \vdash \psi(\vec{x})$.

Image subtoposes and theories of models:

Definition. –
A topos morphism $f : \mathcal{E}' \to \mathcal{E}$ is called a "submersion" if its pull-back component

$$f^* : \mathcal{E} \to \mathcal{E}' \qquad \text{is a faithful functor.}$$

Proposition. –
Any topos morphism $f : \mathcal{E}' \to \mathcal{E}$ uniquely factorises as

$$\mathcal{E}' \xrightarrow{\text{submersion}} \mathrm{Im}(f) \xhookrightarrow{\text{inclusion}} \mathcal{E} \, .$$

Logical interpretation:
If $\mathcal{E} \cong \mathcal{E}_{\mathbb{T}}$ for some geometric theory $\mathbb{T}$
and $f : \mathcal{E}' \to \mathcal{E} \cong \mathcal{E}_{\mathbb{T}}$ corresponds to a $\mathbb{T}$-model M in $\mathcal{E}'$,
the quotient theory $\mathbb{T}'$ of $\mathbb{T}$ which corresponds to $\mathrm{Im}(f) \hookrightarrow \mathcal{E}$
is the "theory of M" in the sense that any implication

$$\varphi(\vec{x}) \vdash \psi(\vec{x}) \, , \quad \vec{x} = (x_1^{E_1}, \cdots, x_n^{E_n})$$

is provable in $\mathbb{T}'$
if and only if it is verified by M, i.e.

$$M\varphi \subseteq M\psi \text{ as subobjects of } ME_1 \times \cdots \times ME_n \text{ in } \mathcal{E}'.$$

Formation of particular descriptions from a general description theory:

- Suppose we consider "elements of reality" $\mathcal{E}_i$, $i \in I$, on which we have partial knowledge incarnated by sites $(\mathcal{C}_i, J_i)$, $i \in I$, necessarily presented by finite data.
- The fact that all $\mathcal{E}_i$'s belong to a natural family should translate into the existence of a joint description geometric theory $\mathbb{T}$ such that each $(\mathcal{C}_i, J_i)$ is endowed with a morphism

$$f_i : \widehat{(\mathcal{C}_i)}_{J_i} \longrightarrow \mathcal{E}_{\mathbb{T}}$$

(corresponding to a model M_i of $\mathbb{T}$ in the topos $\widehat{(\mathcal{C}_i)}_{J_i}$).

Definition. –
In this situation, the quotient theories $\mathbb{T}_i$'s of $\mathbb{T}$, which correspond to the subtoposes

$$\mathrm{Im}(f_i) \hookrightarrow \mathcal{E}_{\mathbb{T}}\,, \quad i \in I\,,$$

can be called the singular descriptions of the elements of reality $\mathcal{E}_i$'s in the joint description formal language $\mathbb{T}$.

Operations on subtoposes:

Theorem. –

(i) *The subtoposes $\mathcal{E}' \hookrightarrow \mathcal{E}$ of a topos $\mathcal{E}$ form an ordered set.*

(ii) *Any family of subtoposes $\mathcal{E}_i \hookrightarrow \mathcal{E}$, $i \in I$, has a union $\bigcup_{i\in I} \mathcal{E}_i \hookrightarrow \mathcal{E}$ and an intersection $\bigcap_{i\in I} \mathcal{E}_i \hookrightarrow \mathcal{E}$.*

(iii) *For any subtopos $\mathcal{E}_1 \hookrightarrow \mathcal{E}$, the map $(\mathcal{E}' \hookrightarrow \mathcal{E}) \mapsto (\mathcal{E}_1 \cup \mathcal{E}' \hookrightarrow \mathcal{E})$ respects arbitrary intersections and, for any $\mathcal{E}_2 \hookrightarrow \mathcal{E}$, there exists a unique subtopos $\mathcal{E}_2 \backslash \mathcal{E}_1 \hookrightarrow \mathcal{E}$ such that $\mathcal{E}_2 \backslash \mathcal{E}_1 \subseteq \mathcal{E}' \Leftrightarrow \mathcal{E}_2 \subseteq \mathcal{E}_1 \cup \mathcal{E}'$.*

Proposition. –

(i) *Any topos morphism $f : \mathcal{E}' \to \mathcal{E}$ induces a push-forward map*

$$f_* : (\mathcal{E}'_1 \hookrightarrow \mathcal{E}') \longmapsto \mathrm{Im}(\mathcal{E}'_1 \hookrightarrow \mathcal{E}' \to \mathcal{E}) = (f_*\mathcal{E}'_1 \hookrightarrow \mathcal{E})$$

which respects arbitrary unions.

(ii) *It also induces a pull-back map* $\quad f^{-1} : (\mathcal{E}_1 \hookrightarrow \mathcal{E}) \longmapsto (f^{-1}\mathcal{E}_1 \hookrightarrow \mathcal{E}')$

characterized by $\quad \mathcal{E}'_1 \subseteq f^{-1}\mathcal{E}_1 \Leftrightarrow f_*\mathcal{E}'_1 \subseteq \mathcal{E}_1$.

(iii) *The map f^{-1} respects arbitrary intersections and finite unions.*

(iv) *If $f : \mathcal{E}' \to \mathcal{E}$ is "essential" (i.e. f^* respects arbitrary limits), the map f^{-1} respects arbitrary unions and there exists a map*

$$f_! : (\mathcal{E}'_1 \hookrightarrow \mathcal{E}') \longmapsto (f_!\mathcal{E}'_1 \hookrightarrow \mathcal{E})$$

characterized by $\quad \mathcal{E}_1 \subseteq f_!\mathcal{E}'_1 \Leftrightarrow f^{-1}\mathcal{E}_1 \subseteq \mathcal{E}'_1$.

Elaboration of a joint description language:

- Start with (mostly unknown) elements of reality $\mathcal{E}_i$, $i \in I$.
  As each $\mathcal{E}_i$ is seen as a (mostly unknown) semantic content,
  it is supposed to have an (unknown) topos structure.
- Partial knowledge on each $\mathcal{E}_i$
  should take the form of a category $\mathcal{C}_i$ (presented by finite data)
  endowed with a functor $\mathcal{C}_i \to \mathcal{E}_i$.
- The fact that all $\mathcal{E}_i$'s belong to a natural family
  should translate into the existence of a joint "description vocabulary" Σ
  and "naming functors"
  $N_i : \mathcal{C}_i \to \mathcal{C}_\Sigma$ (= syntactic category of the theory without axiom Σ).
- Each naming functor N_i defines a topos morphism
  $$\widehat{\mathcal{C}}_i \longrightarrow \widehat{\mathcal{C}}_\Sigma$$
  and any quotient theory $\mathbb{T}$ of Σ defines a subtopos
  $$\mathcal{E}_{\mathbb{T}} \hookrightarrow \widehat{\mathcal{C}}_\Sigma .$$
- The pull-backs of such a $\mathcal{E}_{\mathbb{T}} \hookrightarrow \widehat{\mathcal{C}}_\Sigma$ are subtoposes
  $$\widehat{(\mathcal{C}_i)}_{J_i^{\mathbb{T}}} \hookrightarrow \widehat{\mathcal{C}}_i$$
  where $J_i^{\mathbb{T}}$ = topology = "extrapolation principle" = "interpretation"
  on $\mathcal{C}_i$ derived from the language $\mathbb{T}$.

A principle for syntactic learning and formalized inductive reasoning:

- Suppose we consider a family of "elements of reality" incarnated by (mostly unknown) toposes $\mathcal{E}_i$, $i \in I$, endowed with "partial knowledge functors"
$$k_i : \mathcal{C}_i \longrightarrow \mathcal{E}_i$$
and "naming functors"
$$N_i : \mathcal{C}_i \longrightarrow \mathcal{C}_\Sigma$$
to the syntactic category $\mathcal{C}_\Sigma$ of some formal vocabulary Σ (without axioms).

Principle. – *We look for a quotient theory $\mathbb{T}$ of Σ such that, if*
$$\left(\widehat{(\mathcal{C}_i)}_{J_i^{\mathbb{T}}} \hookrightarrow \widehat{\mathcal{C}}_i\right) = N_i^{-1}\left(\mathcal{E}_{\mathbb{T}} \hookrightarrow \widehat{\mathcal{C}}_\Sigma\right),$$
each $k_i : \mathcal{C}_i \to \mathcal{E}_i$ induces a topos morphism $\widehat{(\mathcal{C}_i)}_{J_i^{\mathbb{T}}} \longrightarrow \mathcal{E}_i$.

Application:

- For each $k_i : \mathcal{C}_i \to \mathcal{E}_i$, there is a biggest subtopos
$\widehat{(\mathcal{C}_i)}_{J_i} \hookrightarrow \widehat{\mathcal{C}}_i$ such that k_i defines $\widehat{(\mathcal{C}_i)}_{J_i} \longrightarrow \mathcal{E}_i$.
- If $N_i : \mathcal{C}_i \to \mathcal{C}_\Sigma$ is "essential", our principle becomes
$$\mathcal{E}_{\mathbb{T}} \subseteq (N_i)_!\left(\widehat{(\mathcal{C}_i)}_{J_i} \hookrightarrow \widehat{\mathcal{C}}_i\right).$$

Morphisms for information extraction:

- Suppose we are considering "elements of reality" $\mathcal{E}_i$, $i \in I$, and partial knowledge on them incarnated by topos morphisms
$$k_i : \widehat{(\mathcal{C}_i)}_{J_i} \longrightarrow \mathcal{E}_i$$
and
$$N_i : \widehat{(\mathcal{C}_i)}_{J_i} \longrightarrow \mathcal{E}_{\mathbb{T}}$$
for a joint description formal language $\mathbb{T}$.
- For any i, $\mathrm{Im}(N_i) \hookrightarrow \mathcal{E}_{\mathbb{T}}$ corresponds to a quotient theory $\mathbb{T}_i$ of $\mathbb{T}$ which can be called a description of $\mathcal{E}_i$ in the language $\mathbb{T}$.
- Suppose we would want to extract from the family
$$\mathrm{Im}(N_i) = \mathcal{E}_{\mathbb{T}_i} \hookrightarrow \mathcal{E}_{\mathbb{T}}, \quad i \in I,$$
some type of information phrased in a language $\mathbb{T}'$.

Proposed geometric form of information extraction:

Information extraction could take the form of a topos morphism
$$f : \mathcal{E}_{\mathbb{T}} \longrightarrow \mathcal{E}_{\mathbb{T}'}.$$
Indeed, it would transform any
$$\mathrm{Im}(N_i) = \mathcal{E}_{\mathbb{T}_i} \hookrightarrow \mathcal{E}_{\mathbb{T}}$$
into a subtopos
$$f_*\mathrm{Im}(N_i) = f_*\mathcal{E}_{\mathbb{T}_i} = \mathcal{E}_{\mathbb{T}'_i} \hookrightarrow \mathcal{E}_{\mathbb{T}'}.$$

Correspondences of information extraction:

Definition. –

(i) *Any pair of toposes* $\mathcal{E}_1, \mathcal{E}_2$ *defines a product topos* $\mathcal{E}_1 \times \mathcal{E}_2$ *characterized by* $\mathrm{Geom}(\mathcal{E}', \mathcal{E}_1 \times \mathcal{E}_2) = \mathrm{Geom}(\mathcal{E}', \mathcal{E}_1) \times \mathrm{Geom}(\mathcal{E}', \mathcal{E}_2), \forall\, \mathcal{E}'$.

(ii) *A correspondence between* $\mathcal{E}_1$ *and* $\mathcal{E}_2$ *is a subtopos*

$$\mathcal{E}_\Gamma \longrightarrow \mathcal{E}_1 \times \mathcal{E}_2.$$

Remarks:

(i) If $\mathcal{E}_1 = \mathcal{E}_{\mathbb{T}_1}$ and $\mathcal{E}_2 = \mathcal{E}_{\mathbb{T}_2}$,
$\mathcal{E}_1 \times \mathcal{E}_2$ is the "classifying topos" of the theory $\mathbb{T}_1 \coprod \mathbb{T}_2$
and correspondences $\mathcal{E}_\Gamma \hookrightarrow \mathcal{E}_1 \times \mathcal{E}_2$
correspond to quotient theories Γ of $\mathbb{T}_1 \coprod \mathbb{T}_2$.

(ii) Any such correspondence $\mathcal{E}_\Gamma \hookrightarrow \mathcal{E}_1 \times \mathcal{E}_2$
transforms subtoposes $(\mathcal{E}_1' \hookrightarrow \mathcal{E}_1)$ into subtoposes $(\mathcal{E}_2' \hookrightarrow \mathcal{E}_2)$
by $(\mathcal{E}_2' \hookrightarrow \mathcal{E}_2) = (\mathrm{pr}_2)_*(\mathcal{E}_\Gamma \cap \mathrm{pr}_1^*(\mathcal{E}_1' \hookrightarrow \mathcal{E}_1))$.

Application to information extraction:

We are looking for quotient theories Γ *of* $\mathbb{T} \coprod \mathbb{T}'$
which transform subtoposes $\mathcal{E}_{\mathbb{T}_i} \hookrightarrow \mathcal{E}_{\mathbb{T}}$, $i \in I$, *into subtoposes* $\Gamma_* \mathcal{E}_{\mathbb{T}_i} \hookrightarrow \mathcal{E}_{\mathbb{T}'}$.

Remark: Conditions $\Gamma_* \mathcal{E}_{\mathbb{T}_i} \subseteq \mathcal{E}_{\mathbb{T}_i'}$, $i \in I$, would mean

$$\mathcal{E}_\Gamma \cap \mathrm{pr}_1^* \mathcal{E}_{\mathbb{T}_i} \subseteq \mathrm{pr}_2^* \mathcal{E}_{\mathbb{T}_i'} \quad , \quad \forall\, i \in I.$$

Constructing morphisms or correspondences by composition:

- A morphism of information extraction
$$f : \mathcal{E}_{\mathbb{T}} \longrightarrow \mathcal{E}_{\mathbb{T}'}$$
could be constructed by composing simpler morphisms
$$\mathcal{E}_{\mathbb{T}_0} \xrightarrow{f_1} \mathcal{E}_{\mathbb{T}_1} \xrightarrow{f_2} \cdots \xrightarrow{f_r} \mathcal{E}_{\mathbb{T}_r}$$
with $\mathbb{T}_0 = \mathbb{T}$, $\mathbb{T}_r = \mathbb{T}'$ and intermediate theories $\mathbb{T}_\alpha$, $1 \leq \alpha < r$.
- Each $f_\alpha : \mathcal{E}_{\mathbb{T}_{\alpha-1}} \to \mathcal{E}_{\mathbb{T}_\alpha}$
could be induced by a "syntactic functor"
$$f_\alpha^* : \mathcal{C}_{\mathbb{T}_\alpha} \longrightarrow \mathcal{C}_{\mathbb{T}_{\alpha-1}}$$
which would express the vocabulary of $\mathbb{T}_\alpha$
in terms of formulas of $\mathbb{T}_{\alpha-1}$.
In other words, it would introduce new concepts
in terms of the language available at the previous step.
- In the same way, a correspondence of information extraction
$$\mathcal{E}_\Gamma \hookrightarrow \mathcal{E}_{\mathbb{T}} \times \mathcal{E}_{\mathbb{T}'}$$
could be constructed as a composite of correspondences
$$\mathcal{E}_{\Gamma_1} \hookrightarrow \mathcal{E}_{\mathbb{T}} \times \mathcal{E}_{\mathbb{T}_1}, \mathcal{E}_{\Gamma_2} \hookrightarrow \mathcal{E}_{\mathbb{T}_1} \times \mathcal{E}_{\mathbb{T}_2}, \cdots, \mathcal{E}_{\Gamma_r} \hookrightarrow \mathcal{E}_{\mathbb{T}_{r-1}} \times \mathcal{E}_{\mathbb{T}'}.$$