Some topos-theoretic principles for syntactic learning

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Tying syntax and semantics:

- Starting remarks:
  → Computers
  store finite sequences of symbols
  and process them using formal rules.
  → In other words, computers are syntactic machines,
  dealing with formal languages.
- Problem:
  → Syntax has no structure and no directions. It can go anywhere.
  → So, syntax needs a guide.
- The experience of natural and mathematical languages:
  → Words are spoken and written in an highly structured way,
  in order to mean something, to make sense.
  → For mathematicians, formal languages describe mathematical structures.
  But, in order to discover these structures, study them
  and derive properties in a formal way,
  they need to imagine and consider “concrete” instances of these structures,
  i.e. “models”.
  → Tarski has defined “syntax” as formalization of “theories”,
  and “semantics” as their incarnation in “models”.

Semantics and topology:

- Experience and intuitions of mathematicians:

  → Mathematicians experience that what they can really understand in mathematics is geometry and topology (the science of forms).

  → For instance, number theory is especially difficult because it is a priori not geometric.
  But huge progress was made in number theory when, thanks to abstraction, it could be studied with topological ideas.

  → From the 1960's to present, different mathematicians and theoretical computer scientists expressed the idea that "semantic information" should incarnate in "topological forms".

- Topology as the science of toposes:

  → Around 1960, Grothendieck introduced the notion of "topos" as the most general mathematical concept of "space" which still allows to define and study

  - points and continuous maps,
  - subspaces, their intersections and their unions,
  - images and pull-backs of subspaces by continuous maps,
  - global invariants (as cohomology and homotopy).

Semantics and toposes:

- A double theorem established in the 1970's shows:
  $\rightarrow$ There is a very general notion of "formalized language",
  the notion of "first-order geometric theory" $\mathbb{T}$,
  such that, for any such theory $\mathbb{T}$,
  its semantics (in the sense of Tarski)
  is incarnated by a (unique) topos $\mathcal{E}_{\mathbb{T}}$:

$$\begin{array}{ccc} \text{models of } \mathbb{T} & \longleftrightarrow & \text{points of } \mathcal{E}, \\ \left\{ \begin{array}{c} \text{continuous families} \\ \text{of models indexed} \\ \text{by a topos } \mathcal{E} \end{array} \right\} & \longleftrightarrow & \left\{ \begin{array}{c} \text{geometric maps} \\ \text{of toposes} \\ \mathcal{E} \rightarrow \mathcal{E}_{\mathbb{T}} \end{array} \right\}. \end{array}$$

  $\rightarrow$ Conversely, any topos $\mathcal{E}$ incarnates the semantics
  of infinitely many such theories $\mathbb{T}$, i.e. $\mathcal{E} \cong \mathcal{E}_{\mathbb{T}}$.
- A semantic principle for guiding syntactic processing:
  Work with theories $\mathbb{T}$,
  only using processes which are defined
  in terms of associated toposes $\mathcal{E}_{\mathbb{T}}$
  and geometric maps $\mathcal{E}' \rightarrow \mathcal{E}$ relating them.

Elaboration of appropriate description languages:

- Before even thinking about how to process representations of some type of elements of reality (e.g. images), it is needed to elaborate an appropriate formal description language $\mathbb{T}$.
  $\rightarrow$ It seems that this preliminary step is often overlooked,
  whereas it is most important.
- Such a formal description language necessarily consists in a vocabulary Σ and a list of axioms (= grammar rules).
- The vocabulary has to be chosen so that
  - it is rich enough to allow the expression of distinctions and differences,
  - it is not too rich, so that it expresses analogies and shared identities, and naturally leads to the discovery of natural laws.
- The axioms (expressing natural laws) should be infered from particular instances, thanks to the chosen vocabulary.

Axiom systems and topology:

- A theorem (established in 2008 by O. Caramello) shows that any equivalence of toposes $\mathcal{E} \cong \mathcal{E}_{\mathbb{T}}$ induces a one-to-one correspondence

$$\left\{\begin{array}{c}\text{subtoposes}\\ \mathcal{E}_1 \hookrightarrow \mathcal{E}\end{array}\right\} \longleftrightarrow \left\{\begin{array}{c}\text{geometric theories } \mathbb{T}_1\\ \text{deduced from } \mathbb{T} \text{ by}\\ \text{adding new axioms}\\ \text{(in the same vocabulary)}\end{array}\right\}.$$

- In particular, adding axioms to a vocabulary Σ to define a theory $\mathbb{T}$ amonts to define a subtopos $\mathcal{E}_{\mathbb{T}} \hookrightarrow \mathcal{E}_{\Sigma}$.
- If $\mathcal{E} \cong \mathcal{E}_{\mathbb{T}}$ and $\mathcal{E}' \cong \mathcal{E}_{\mathbb{T}'}$, any geometric map $f : \mathcal{E}' \longrightarrow \mathcal{E}$ induces two maps

$$\begin{cases} f_* : (\mathcal{E}'_1 \hookrightarrow \mathcal{E}') & \longmapsto \quad (f_*\mathcal{E}'_1 \hookrightarrow \mathcal{E}), \\ f^{-1} : (\mathcal{E}_1 \hookrightarrow \mathcal{E}) & \longmapsto \quad (f^{-1}\mathcal{E}_1 \hookrightarrow \mathcal{E}'), \end{cases}$$

and so two processes of transformation of theories

$$\left\{\begin{array}{c}\text{systems of}\\ \text{extra axioms}\\ \text{added to } \mathbb{T}'\end{array}\right\} \underset{f^{-1}}{\overset{f_*}{\rightleftarrows}} \left\{\begin{array}{c}\text{systems of}\\ \text{extra axioms}\\ \text{added to } \mathbb{T}\end{array}\right\}.$$

Formal application of a description vocabulary:

- Suppose we are given elements of reality $\mathcal{E}_i$, $i \in I$, which we feel belong to a natural family.
- Suppose we are able to partially represent each $\mathcal{E}_i$, by identifying some components and some relations between them, taking the form of a (finitely presented) "category"
$$\mathcal{C}_i$$
(consisting in components, relations between pairs of components and a law of composition of relations).
- The fact that all $\mathcal{E}_i$'s are felt to belong to a family should translate into the definition of a joint vocabulary Σ allowing to give names to the components and relations of the representations $\mathcal{C}_i$'s, in the form of "naming functors"
$$N_i : \mathcal{C}_i \longrightarrow \mathcal{C}_\Sigma$$
to the "syntactic category" $\mathcal{C}_\Sigma$ of Σ, whose components and relations consist in the words of Σ and the sentences in these words.

A geometric process of inductive reasoning:

- We need more geometric facts from topos theory:
  - **(1)** Any (small) category $\mathcal{C}$ defines a topos $\widehat{\mathcal{C}}$.
    The subtoposes $\widehat{\mathcal{C}}_J \hookrightarrow \widehat{\mathcal{C}}$ are indexed by so-called "topologies" J on $\mathcal{C}$.
  - **(2)** Each $\widehat{\mathcal{C}}_J$ can be considered as a completion of $\mathcal{C}$,
    so that the corresponding "topology" J on $\mathcal{C}$ can be called
    an "extrapolation principle" or "interpretation system".
  - **(3)** If Σ is a vocabulary, $\mathcal{E}_\Sigma$ is a subtopos $\mathcal{E}_\Sigma \hookrightarrow \widehat{\mathcal{C}}_\Sigma$
    and systems of extra axioms correspond to smaller subtoposes $\mathcal{E}_{\mathbb{T}} \hookrightarrow \widehat{\mathcal{C}}_\Sigma$.
  - **(4)** Any functor $N : \mathcal{C}' \to \mathcal{C}$ defines a geometric map $\widehat{\mathcal{C}'} \longrightarrow \widehat{\mathcal{C}}$.
- Formal inductive reasoning:

  Naming functors $N_i : \mathcal{C}_i \to \mathcal{C}_\Sigma$ define geometric maps $\widehat{\mathcal{C}}_i \to \widehat{\mathcal{C}}_\Sigma$.
  We look for a minimal system of axioms on the vocabulary Σ, defining

  $$\mathcal{E}_{\mathbb{T}} \longrightarrow \widehat{\mathcal{C}}_\Sigma$$

  such that the subtoposes defined by pull-back $(\widehat{\mathcal{C}}_i)_{J_i} \longrightarrow \widehat{\mathcal{C}}_i$
  are meaningful representations of $\mathcal{E}_i$,

  in the sense that all points of $(\widehat{\mathcal{C}}_i)_{J_i}$
  make sense in the context of each element of reality $\mathcal{E}_i$.

The geometric process of formation of singular descriptions:

- Suppose the previous "formal inductive reasoning" process has allowed to define a system $\mathbb{T}$ of axioms on the vocabulary Σ, defining a subtopos
$$\mathcal{E}_{\mathbb{T}} \hookrightarrow \widehat{\mathcal{C}}_{\Sigma}$$
and, by pull-back through the naming functors $N_i : \mathcal{C}_i \to \mathcal{C}_{\Sigma}$, some "meaningful representations"
$$(\widehat{\mathcal{C}_i})_{J_i} \hookrightarrow \widehat{\mathcal{C}_i}$$
of the elements of reality $\mathcal{E}_i$'s.
- Induced linguistic descriptions:
Each induced geometric map
$$N_i : (\widehat{\mathcal{C}_i})_{J_i} \longrightarrow \mathcal{E}_{\mathbb{T}}$$
has an image which is a subtopos
$$\mathrm{Im}(N_i) = \mathcal{E}_{\mathbb{T}_i} \hookrightarrow \mathcal{E}_{\mathbb{T}}$$
necessarily associated to a theory $\mathbb{T}_i$ deduced from $\mathbb{T}$ by adding a system of extra axioms.
Each such $\mathbb{T}_i$ can be called a particular linguistic description of the element of reality $\mathcal{E}_i$ in the joint description language $\mathbb{T}$.

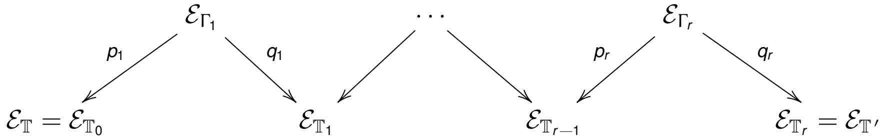
A general form of geometric information extraction:

• Suppose we have elaborated an appropriate formal description language $\mathbb{T}$ for a family of elements of reality $\mathcal{E}_i$, $i \in I$,
represented by subtoposes $\qquad \mathcal{E}_{\mathbb{T}_i} \hookrightarrow \mathcal{E}_{\mathbb{T}}$.

• Suppose we would want to extract from these representations some type of information phrased in a language $\mathbb{T}'$.
So we need a geometric process for transforming

$$\{\text{subtoposes of } \mathcal{E}_{\mathbb{T}}\} \longrightarrow \{\text{subtoposes of } \mathcal{E}_{\mathbb{T}'}\}.$$

• It could take the form of a chain of geometric maps



where, for any r', $1 \leq r' \leq r$,

$$(q_{r'})_* \circ p_{r'}^{-1} \qquad \text{transforms}$$

$$\{\text{subtoposes of } \mathcal{E}_{\mathbb{T}_{r'-1}}\} \longrightarrow \{\text{subtoposes of } \mathcal{E}_{\mathbb{T}_{r'}}\}.$$

• It would be enough to define each theory $\Gamma_{r'}$, $1 \leq r' \leq r$,
by adding a system of extra “correlation axioms”
to the join theory $\qquad \mathbb{T}_{r'-1} \coprod \mathbb{T}_{r'}$.