Relating semantics and approximation through Grothendieck toposes

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From meta-mathematics to mathematics?

The themes of our conference:

- "Learning over topological spaces".
- "Semantic and goal-oriented communication".

A striking common feature:

The words

- "learning",
- "semantic" (in the sense of "conveying meaning"),
- "goal",

make natural sense for human minds,
and in particular apply to the work of mathematicians,
but
they do not belong to classical mathematics,
they do not represent classical mathematical objects.

A derived necessary step before building a theory of "semantic information":

Can meta-mathematical notions as "learning", "semantic" or "goal" be modellized inside mathematics?

Syntax and semantics, according to Tarski:

Tarski has proposed precise formal definitions
for the words “syntax” and “semantics”:

Definition. –

(i) *A mathematical syntax is a theory consisting in*

- *a vocabulary*
  - *names of structures (e.g. group G),*
  - *names of operation*
    (e.g. multiplication $GG \to G$*, inverse* $G \xrightarrow{(\bullet)^{-1}} G$*, unit element* $\to G$*),*
  - *names of relations (e.g. order* $\leq$ *or equivalence* $\sim$*),*
- *a family of axioms phrased in the given vocabulary.*

(ii) *The semantics of a given theory* $\mathbb{T}$
consists in its (set-valued) “models” M, i.e.

- *sets,*
- *maps,*
- *subsets,*

named after the elements of vocabulary of $\mathbb{T}$*, verifying its axioms,*
and related by the maps between these structures
which respect their inner operations and relations.

Topossic semantics:

Basic fact about toposes as "pastiches" of the category of sets:

According to Grothendieck and Giraud,
toposes are categories which are just as good as Set:

- *They have arbitrary products* $\prod_{i \in I} E_i$.
- *They have arbitrary sums* $\coprod_{i \in I} E_i$.
- *Equalizers of pairs of arrows* $E \overset{f}{\underset{g}{\rightrightarrows}} E'$ *are well-defined* $\{f = g\} \hookrightarrow E$.
- *Quotients* $E \twoheadrightarrow E'$ *correspond one-to-one to equivalence relations* $\sim \hookrightarrow E \times E$.
- *Relations* $R \hookrightarrow E \times E$ *generate equivalence relations* $\sim \hookrightarrow E \times E$.
- $\ldots$

Consequences for any theory $\mathbb{T}$:

(i) $\mathbb{T}$ has models in any topos $\mathcal{E}$ as well as in Set.

(ii) If $\mathbb{T}$ is first-order, its models in $\mathcal{E}$ make up a category $\mathbb{T}\text{-mod}\,(\mathcal{E})$.

(iii) If $\mathbb{T}$ is first-order and geometric, topos maps $f : \mathcal{E}' \to \mathcal{E}$ induce "change of parameters" transforms of models

$$f^* : \mathbb{T}\text{-mod}\,(\mathcal{E}) \to \mathbb{T}\text{-mod}\,(\mathcal{E}').$$

Topossic incarnation of semantics:

Theorem (Grothendieck, Hakim, Lawvere, Joyal, Makkai, Reyes, $\cdots$). –
For any theory $\mathbb{T}$ which is first-order and geometric,
there exists a topos $\mathcal{E}_{\mathbb{T}}$ (unique up to equivalence) such that:
- *For any topos $\mathcal{E}$, there is a natural equivalence:*

$$\begin{array}{rcl} \mathbb{T}\text{-mod}\,(\mathcal{E}) & \xleftarrow{\sim} & \mathrm{Geom}\,(\mathcal{E}, \mathcal{E}_{\mathbb{T}}) \\ \{\mathcal{E}\text{-valued } \mathbb{T}\text{-models}\} = & & = \{\text{topos maps } \mathcal{E} \longrightarrow \mathcal{E}_{\mathbb{T}}\} \\ \{\mathcal{E}\text{-parametrized } \mathbb{T}\text{-models}\} = & & = \{\mathcal{E}\text{-parametrized points of } \mathcal{E}_{\mathbb{T}}\} \end{array}$$

- *For any topos map $\mathcal{E}' \xrightarrow{e} \mathcal{E}$,*

$$\left.\begin{array}{c} \text{the induced} \\ \text{“change of parameters” transform} \\ e^* : \mathbb{T}\text{-mod}\,(\mathcal{E}) \longrightarrow \mathbb{T}\text{-mod}\,(\mathcal{E}') \end{array}\right\} \text{ corresponds to } \left\{\begin{array}{c} \text{composition with } e \\ (\mathcal{E} \xrightarrow{m} \mathcal{E}_{\mathbb{T}}) \mapsto (\mathcal{E}' \xrightarrow{e} \mathcal{E} \xrightarrow{m} \mathcal{E}_{\mathbb{T}}). \end{array}\right.$$

Remarks:
- Topological spaces X define toposes $\mathcal{E}_X$,

and continuous maps $X' \xrightarrow{x} X$ define topos maps $\mathcal{E}_{X'} \longrightarrow \mathcal{E}_X$.
- In particular, the one-point space $\{\bullet\}$ defines the topos Set,

and elements $x \in X$ of topological spaces X define topos maps Set $\longrightarrow \mathcal{E}_X$.
- For this reason, a “point” of a topos $\mathcal{E}$ is by definition a topos map Set $\longrightarrow \mathcal{E}$

and any topos map $\mathcal{E}' \to \mathcal{E}$
is called a $\mathcal{E}'$-parametrized (generalized) point of $\mathcal{E}$.

Geometric expressions of the semantics of theories:

• On the one hand, models of first-order geometric theories $\mathbb{T}$ correspond to points of the associated toposes $\mathcal{E}_{\mathbb{T}}$.
• On the other hand, toposes $\mathcal{E}$ admit by definition presentations as categories of set-valued sheaves $\mathcal{E} \cong \widehat{\mathcal{C}}_J$ on small categories $\mathcal{C}$ endowed with Grothendieck topologies J.

Theorem (Grothendieck, SGA 4). –
Consider a presentation of a topos $\mathcal{E} \cong \widehat{\mathcal{C}}_J$
by a small category $\mathcal{C}$ *with well-defined*

$$\begin{cases} - \quad \textit{finite products} \quad U_1 \times \cdots \times U_n, \\ - \quad \textit{equalizers} \quad \mathrm{eq}\,(U \underset{g}{\overset{f}{\rightrightarrows}} V) = \{f = g\} \hookrightarrow U. \end{cases}$$

Then points of $\mathcal{E}$ $\quad \mathrm{Set} \to \mathcal{E}$ $\quad$ *(or generalized points* $\mathcal{E}' \to \mathcal{E}$*)*
correspond to "functors" $\quad \rho : \mathcal{C} \to \mathrm{Set}$ $\quad$ *(or* $\rho : \mathcal{C} \to \mathcal{E}'$*)*
which
- respect finite products and equalizers,
- transform J-covering families $(U_i \xrightarrow{u_i} U)_{i \in I}$
into globally surjective (or globally epimorphic) families

$$(\rho(U_i) \xrightarrow{\rho(u_i)} \rho(U))_{i \in I} \quad \textit{in} \quad \mathrm{Set} \quad \textit{(or in} \quad \mathcal{E}'\textit{)}.$$

Understanding Grothendieck's theorem as an approximation theorem:

Let's make explicit Grothendieck's theorem in the particular case of topological spaces:

Particular case of the previous theorem. –

Let X be a topological space. Let $\mathcal{C} \subseteq \mathcal{C}_X$ be a set of open subsets of X such that

$$\begin{cases} - & \mathcal{C} \text{ is stable under finite intersection,} \\ - & \mathcal{C} \text{ is "dense" i.e. any open subset of } X \text{ is a union of elements of } \mathcal{C}. \end{cases}$$

Then points of $\mathcal{E}_X$

$$\mathrm{Set} \to \mathcal{E}_X \qquad \text{(or generalized points } \mathcal{E}_{X'} \to \mathcal{E}_X\text{)}$$

correspond to maps

$$\rho : \mathcal{C} \to \{\text{subsets of } \{\bullet\}\} \qquad \text{(or } \rho : \mathcal{C} \to \{\text{open subsets of } X'\}\text{)}$$

which

$$\begin{cases} - & \text{respect finite intersections,} \\ - & \text{transform covering families} \\ & \text{into globally surjective families.} \end{cases}$$

Example:

If $X = \mathbb{R}$, one may take $\mathcal{C}_X = \{\text{intervals }]m, M[, \quad m, M \in Q\}$ for any dense subset $Q \subseteq \mathbb{Q}$.

$\Rightarrow$ A real number is a family of compatible answers to the questions:

$$\begin{cases} \text{For any interval }]m, M[, \quad m, M \in Q, \\ \text{does it belong to this interval or not?} \end{cases}$$

Approximation and learning for the semantics of theories:

As consequences of Grothendieck's theorem, one may state and propose:

Corollary. –
Consider presentations $\mathcal{E} \cong \widehat{\mathcal{C}}_J$ *of a given topos* $\mathcal{E}$.
Then functors $\rho : \mathcal{C} \to \mathrm{Set}$ *(or* $\rho : \mathcal{C} \to \mathcal{E}'$*) which*

- *respect finite products and equalizers,*
- *transform* J*-covering families into globally surjective families*

do not depend on the choice of $(\mathcal{C}, J)$.

Definition. – *An approximation of a point* $\mathrm{Set} \to \mathcal{E}$ *or* $\mathcal{E}' \to \mathcal{E}$
seen as such a functor $\rho : \mathcal{C} \to \mathrm{Set}$ *or* $\rho : \mathcal{C} \to \mathcal{E}'$
is defined as the restriction of ρ *to a finite subdiagram* D *of* $\mathcal{C}$.

Remarks:

- Replacing D by a bigger finite diagram D' represents "learning".
- The "goal" of the learning process is the full $\rho : \mathcal{C} \to \mathrm{Set}$ or $\rho : \mathcal{C} \to \mathcal{E}'$.
  It is an ideal goal which, in general, cannot be reached.
- If $\mathcal{E} \cong \mathcal{E}_{\mathbb{T}}$ and $\mathbb{T}$ has a finite vocabulary,
  one may choose finite subdiagrams $D \hookrightarrow \mathcal{C}$
  such as the functors ρ are fully determined by their restrictions to D.

When can a point be considered concrete?

The notion of point of a topos $\mathcal{E}$
(or even of a topological space X, or even of $X = \mathbb{R}$)
is abstract and ideal: only very particular points can be made fully explicit!

Proposed definition of concreteness. –
Consider a topos $\mathcal{E}$ presented as $\mathcal{E} \cong \widehat{\mathcal{C}}_J$.
Then a point $\mathrm{Set} \to \mathcal{E}$ (or $\mathcal{E}' \to \mathcal{E}$) seen as a functor

$$\rho : \mathcal{C} \to \mathrm{Set} \qquad \textit{(or } \rho : \mathcal{C} \to \mathcal{E}'\textit{)}$$

can be called "$\mathcal{C}$-concrete" if:

- (1) *The functor ρ is uniquely determined by its restriction ρ_D to a finite subdiagram $D \hookrightarrow \mathcal{C}$.*
- (2) *For any bigger finite subdiagram $D' \hookrightarrow \mathcal{C}$, there is an algorithm allowing to compute from ρ_D, in finitely many steps, the restriction $\rho_{D'}$ of ρ to D'.*

Example: If $\mathbb{T}$ is an algebraic theory in a finite vocabulary,
models of $\mathbb{T}$ with values in finite sets
are "concrete points" of $\mathcal{E}_{\mathbb{T}}$ in this sense
with respect to the "cartesian syntactic category" $\mathcal{C} = \mathcal{C}_{\mathbb{T}}^{\mathrm{cart}}$.

Internal traces of points:

Unlike in set-based mathematics,
points (or generalized points) of a topos $\mathcal{E}$ are external,
as they are defined as topos maps $\mathrm{Set} \to \mathcal{E}$ (or $\mathcal{E}' \to \mathcal{E}$),
just as models of a theory $\mathbb{T}$ are external to $\mathbb{T}$.
Nevertheless, points of toposes have internal traces defined as subtoposes:

Proposition. – *There are well-defined notions of*

- *surjective map of toposes,*
- *embedding of a subtopos into a topos,*

such that any (generalized) point of a topos $\mathcal{E}$

$e : \mathcal{E}' \to \mathcal{E}$ *uniquely factorizes as:*

$$\mathcal{E}' \xrightarrow[\textit{surjective map}]{} \mathrm{Im}(e) \xhookrightarrow[\textit{subtopos embedding}]{} \mathcal{E}$$

Remarks:

- As a corollary, any topos map $\mathcal{E}' \xrightarrow{e} \mathcal{E}$ induces a push-forward operation

$$e_* : \{\text{subtoposes of } \mathcal{E}'\} \longrightarrow \{\text{subtoposes of } \mathcal{E}\}$$
$$(\mathcal{E}'_1 \hookrightarrow \mathcal{E}') \longmapsto (\mathrm{Im}(\mathcal{E}'_1 \hookrightarrow \mathcal{E}' \xrightarrow{e} \mathcal{E}) = e_*\mathcal{E}'_1 \hookrightarrow \mathcal{E}).$$

- It also induces a pull-back operation

$$e^{-1} : \{\text{subtoposes of } \mathcal{E}\} \to \{\text{subtoposes of } \mathcal{E}'\}$$

characterized by $\quad e^{-1}\mathcal{E}_1 \supseteq \mathcal{E}'_1 \iff \mathcal{E}_1 \supseteq e_*(\mathcal{E}'_1)\,.$

Topological expressions of traces of points:

Theorem (Grothendieck, SGA 4). –
Let $\mathcal{E}$ be a topos presented as the category of "sheaves"

$$\widehat{\mathcal{C}}_J \quad \text{on a small category } \mathcal{C} \text{ endowed with a topology } J.$$

Then there is a one-to-one correspondence

$$\begin{array}{ccc} K & \longmapsto & \widehat{\mathcal{C}}_K \hookrightarrow \widehat{\mathcal{C}}_J \cong \mathcal{E}\,, \\ \left\{ \begin{array}{c} \text{topologies } K \text{ on } \mathcal{C} \\ \text{which contain } J \end{array} \right\} & \xrightarrow{\sim} & \{\text{subtoposes of } \widehat{\mathcal{C}}_J \cong \mathcal{E}\}. \end{array}$$

Reminder:
A topology J on a category $\mathcal{C}$ is a notion of "covering" families

$$(X_i \xrightarrow{x_i} X)_{i \in I} \quad \text{of arrows of } \mathcal{C}$$

such that:

- **(0)** Any family $(X_i \xrightarrow{x_i} X)_{i \in I}$ whose associated "sieve"
$$\{X' \xrightarrow{x} X \mid x \text{ factorizes as } X' \to X_i \xrightarrow{x_i} X \text{ for at least one } i \in I\}$$
contains a covering family, is itself covering.
- **(1)** For any X, $X \xrightarrow{\mathrm{id}_X} X$ is covering.
- **(2)** Any arrow $X' \to X$ transforms by pull-back coverings of X into coverings of X'.
- **(3)** A family $(X_i \xrightarrow{x_i} X)_{i \in I}$ is a covering if it is transformed into coverings by pull-back along the elements $X'_j \xrightarrow{x'_j} X$ of a covering family of X.

Approximations of topologies:

Example:

- Let X be a topological space, and $\mathcal{C}$ a dense family of open subsets of X.
  An approximation of a subspace $X' \hookrightarrow X$ on some $U \in \mathcal{C}$
  is a family of smaller open subsets $\quad U_i \subseteq U,\ U_i \in \mathcal{C},\ i \in I$,
  such that $\quad X' \cap U \subseteq \bigcup_{i \in I} U_i$.
- This is equivalent to requesting that $\quad (U_i \hookrightarrow U)_{i \in I} \quad$ is covering
  for the topology $J_{X'}$ on $\mathcal{C}$ corresponding to the subtopos $\quad \mathcal{E}_{X'} \hookrightarrow \mathcal{E}_X \cong \widehat{(\mathcal{C})}_{J_X}$.

Proposition. –
Consider a topos map $\quad e : \mathcal{E}' \longrightarrow \mathcal{E} \cong \widehat{\mathcal{C}}_J \quad$ *seen as a functor* $\quad \rho : \mathcal{C} \longrightarrow \mathcal{E}'$.
Then a family of arrows of $\mathcal{C}$

$$(X_i \xrightarrow{x_i} X)_{i \in I}$$

is covering for the topology $K_e \supseteq J$ *corresponding to* $\mathrm{Im}(e) \hookrightarrow \mathcal{E} \cong \widehat{\mathcal{C}}_J$

if and only if the family $\quad (\rho(X_i) \xrightarrow{\rho(x_i)} \rho(X))_{i \in I} \quad$ *is globally epimorphic in* $\mathcal{E}'$.

Consequence for approximating topologies:

- If we only know an approximation of ρ, consisting in a restriction
  $\rho_D : D \longrightarrow \mathcal{E}' \quad$ to a subdiagram $\quad D \hookrightarrow \mathcal{C}$,
  then we can only deduce the restriction of K_e to families of arrows of D.
- In some cases, it may be enough to generate the whole topology K_e.

Logical expressions of traces of points:

Theorem (Caramello, PhD thesis and [Theories, Sites, Toposes]). –
Let $\mathcal{E}$ be a topos incarnating the semantics of a first-order geometric theory $\mathbb{T}$

$$\mathcal{E} \cong \mathcal{E}_{\mathbb{T}}.$$

Then there is a one-to-one correspondence

$$\begin{array}{ccc} \mathbb{T}' & \longmapsto & \mathcal{E}_{\mathbb{T}'} \hookrightarrow \mathcal{E}_{\mathbb{T}} \cong \mathcal{E}, \\ \left\{ \begin{array}{c} \text{“quotient” theories } \mathbb{T}' \text{ of } \mathbb{T}, \\ \text{considered up to} \\ \text{syntactic equivalence} \end{array} \right\} & \longrightarrow & \{\text{subtoposes of } \mathcal{E}_{\mathbb{T}} \cong \mathcal{E}\}. \end{array}$$

Reminder:

- A “quotient” of a theory $\mathbb{T}$
  is a theory $\mathbb{T}'$ written in the same vocabulary
  whose list of axioms contains the axioms of $\mathbb{T}$.
- Two quotient theories $\mathbb{T}_1$ and $\mathbb{T}_2$ of $\mathbb{T}$
  are “syntactically equivalent” if
  the axioms of $\mathbb{T}_1$ or $\mathbb{T}_2$
  are provable from the axioms of the other theory.

Approximation of theories:

Proposition. –
Consider a topos map $e : \mathcal{E}' \to \mathcal{E} \cong \mathcal{E}_{\mathbb{T}}$
corresponding to a model $M \in \mathbb{T}\text{-mod}(\mathcal{E}')$ *of a theory* $\mathbb{T}$ *in a topos* $\mathcal{E}'$.
Then an implication between formulas written in the language of $\mathbb{T}$

$$\varphi(x_1^{A_1}, \cdots, x_n^{A_n}) \vdash \psi(x_1^{A_1}, \cdots, x_n^{A_n})$$

is provable in the quotient theory $\mathbb{T}_e$ *of* $\mathbb{T}$ *corresponding to*

$$\mathrm{Im}(e) \hookrightarrow \mathcal{E} \cong \mathcal{E}_{\mathbb{T}}$$

if and only if the interpretations of φ *and* ψ *in the model* M

$$M\varphi, M\psi \hookrightarrow MA_1 \times \cdots \times MA_n$$

verify as subobjects $M\varphi \subseteq M\psi$.

Consequence for approximating theories:

- If $\mathcal{E} \cong \mathcal{E}_{\mathbb{T}}$ is presented as $\mathcal{E} \cong \widehat{\mathcal{C}}_J$, and the topos map $e : \mathcal{E}' \to \mathcal{E}$ corresponds to a functor $\rho : \mathcal{C} \to \mathcal{E}'$ whose restriction $\rho_D : D \to \mathcal{E}'$ to a subdiagram $D \hookrightarrow \mathcal{C}$ we only know, then we can only deduce $\mathbb{T}_e$-provable implications between formulas φ, ψ which are interpretable in terms of $\rho_D : D \to \mathcal{E}'$.
- In some cases, it may be enough to provide a list of axioms of $\mathbb{T}_e$.

A general scheme of geometric processing and approximations:

- Classically, data are represented as points of (high dimension linear) spaces. But, in the wider context of toposes, it would seem more natural to represent data as subtoposes of some topos.
- Then subtoposes could be processed geometrically by composing transforms of the form

$$q_* \circ p^{-1} : \{\text{subtoposes of } \widehat{\mathcal{C}}_J\} \longrightarrow \{\text{subtoposes of } \widehat{\mathcal{D}}_K\}$$

defined by topos maps

$$\begin{array}{ccccc} & & \mathcal{E} & & \\ & \swarrow^{p} & & \searrow^{q} & \\ \widehat{\mathcal{C}}_J & & & & \widehat{\mathcal{D}}_K \end{array}$$

corresponding to functors $\rho : \mathcal{C} \longrightarrow \mathcal{E}$, $\sigma : \mathcal{D} \longrightarrow \mathcal{E}$.

Proposition. – *Suppose $\mathcal{E} \cong \widehat{\mathcal{L}}_L$ and ρ, c are lifted to $\rho : \mathcal{C} \longrightarrow \mathcal{L}$, $\sigma : \mathcal{D} \longrightarrow \mathcal{L}$.*
(i) For any subtopos $\mathcal{E}_1 \hookrightarrow \widehat{\mathcal{C}}_J$ corresponding to a topology $J_1 \supseteq J$ on $\mathcal{C}$, its pull-back $p^{-1}\mathcal{E}_1 \hookrightarrow \widehat{\mathcal{L}}_L$ corresponds to the topology $L_1 \supseteq L$ on $\mathcal{L}$ generated by L and the transforms by ρ of the J_1-covering families of $\mathcal{C}$.
(ii) The push-forward $q_* \circ p^{-1}\mathcal{E}_1 \hookrightarrow \widehat{\mathcal{D}}_K$ corresponds to the topology $K_1 \supseteq K$ on $\mathcal{D}$ for which a family of arrows of $\mathcal{D}$ is covering if and only if its transform by σ is L_1-covering.

Remark: It makes sense to restrict all data and computations to subdiagrams of $\mathcal{C}$ and $\mathcal{D}$.

Semantic intertwinning for goal-oriented processing?

Question. – *If data are already represented as subtoposes of a topos*

$\mathcal{E}_{\mathbb{T}_0}$ *incarnating the semantics of a theory* $\mathbb{T}_0$*,*

how could we elaborate from $\mathbb{T}_0$ *another theory* $\mathbb{T}_1$ *related to* $\mathbb{T}_0$ *by topos maps*

$$\mathcal{E}_{\mathbb{T}_0} \xleftarrow{\;p\;} \mathcal{E} \xrightarrow{\;q\;} \mathcal{E}_{\mathbb{T}_1}$$

corresponding to a double model structure of types $\mathbb{T}_0$ *and* $\mathbb{T}_1$ *on a topos* $\mathcal{E}$*?*

Expectation:

- The language of $\mathbb{T}_1$ should be more appropriate than the language of $\mathbb{T}_0$ to the type of information on the data we are interested in.
- Most often, the language of $\mathbb{T}_1$ should be "more global".

Principles:

- The fact that the language of $\mathbb{T}_1$ is "more global" could correspond to the fact that it would apply to an "invariant" construction on the topos $\mathcal{E}$.
- This invariant construction would be interpreted as a model of $\mathbb{T}_1$ in $\mathcal{E}$.
- This invariant construction could be higher-order,

  and as a consequence non-compatible with pull-backs by topos maps $\mathcal{E}' \to \mathcal{E}$.
- This would make the choice of the model $\mathcal{E} \to \mathcal{E}_{\mathbb{T}_0}$ very important.

  It should be chosen so as to maximize the intertwinning of $\mathcal{E}_{\mathbb{T}_0}$ and $\mathcal{E}_{\mathbb{T}_1}$.