Towards a geometric theory of cohomology functors: the case of degree 0

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Looking for a theory of cohomology functors:

Conjecture (O. Caramello). –
For any "good" geometric category $\mathcal{G}$,
there should exist a (first-order geometric) theory $\mathbb{T}$
of cohomology functors on objects (or pairs of objects) of $\mathcal{C}$

$$\begin{aligned} (X, i) &\longmapsto H^i(X)\,, \\ (Z \hookrightarrow X, i) &\longmapsto H^i(X, Z) \end{aligned}$$

with coefficients in a (not predetermined) field $K \supseteq \mathbb{Q}$,
such that:

(1) *All classical cohomology functors (such as singular cohomology, or ℓ-adic cohomology of algebraic varieties) should appear as models of this theory $\mathbb{T}$.*

(2) *The "classifying topos" $\mathcal{E}_{\mathbb{T}}$ of $\mathbb{T}$ should be "atomic" and "2-valued", implying $\mathbb{T}$ is a "complete" theory and all its models share the same properties (in particular the same dimensions over coefficient fields).*

A few words about the theory of "classifying toposes":

Theorem (Hakim, Grothendieck, Lawvere, Joyal, Makkai, Reyes, $\cdots$). –
*Let $\mathbb{T}$ be a first-order theory which is "geometric", meaning that the formalization of its axioms only uses the logical symbols $=$, $\wedge$ (finite conjunction), $\top$ (true), $\bigvee$ (arbitrary disjunction), $\bot$ (false), $\exists$ (existential quantifier).
Then the 2-functor on the 2-category of toposes*

$$\begin{aligned} \text{topos } \mathcal{E} &\longmapsto \mathbb{T}\text{-mod}(\mathcal{E}) = \{\textit{category of } \mathcal{E}\textit{-valued models of } \mathbb{T}\}, \\ (\mathcal{E}' \xrightarrow{(f^*, f_*)} \mathcal{E}) &\longmapsto (f^* : \mathbb{T}\text{-mod}(\mathcal{E}) \to \mathbb{T}\text{-mod}(\mathcal{E}')) \end{aligned}$$

is representable by a topos $\mathcal{E}_{\mathbb{T}}$ (which is uniquely determined up to equivalence), meaning that, for any topos $\mathcal{E}$,

$$\mathbb{T}\text{-mod}(\mathcal{E}) \xrightarrow{\approx} \mathrm{Geom}(\mathcal{E}, \mathcal{E}_{\mathbb{T}}) = \{\textit{category of topos morphism } \mathcal{E} \to \mathcal{E}_{\mathbb{T}}\}$$

and, in particular,

$$\mathbb{T}\text{-mod}(\mathrm{Set}) \xrightarrow{\approx} \mathrm{pt}(\mathcal{E}_{\mathbb{T}}) = \{\textit{category of "points" of } \mathcal{E}_{\mathbb{T}}\}.$$

Atomic toposes:

Definition. –

(i) *An object E of a topos $\mathcal{E}$ is called an "atom" if the only subobjects of E are E and $\emptyset$.*

(ii) *A topos $\mathcal{E}$ is called "atomic" if any object of $\mathcal{E}$ decomposes as a disjoint sum of "atoms".*

(iii) *A topos $\mathcal{E}$ is called "2-valued" if its terminal object 1 is an atom.*

Example. – For any (topological) group G, its classifying topos

$$\text{BG} = \{\text{category of (continuous) actions of } G\}$$

is atomic and 2-valued. Its atoms are transitive actions.

Proposition. –
A first-order geometric theory $\mathbb{T}$ is "complete" (in the sense that any "geometric" formula φ written in the language of $\mathbb{T}$ without "free" variables is either provably true or provably false) if and only if its classifying topos $\mathcal{E}_{\mathbb{T}}$ is "2-valued".

Representation of atomic toposes:

Proposition. –

(i) If $\mathcal{E}$ is an "atomic" topos,
its full subcategory $\mathcal{E}^{\mathrm{at}}$ of "atoms" is essentially small
and $\mathcal{E}$ identifies with the topos of sheaves on the site $(\mathcal{E}^{\mathrm{at}}, J_{\mathrm{at}})$
where J_{at} is the "atomic topology" for which
covering families $(X_i \to X)_{i \in I}$ are exactly non-empty families.

(ii) More generally, a functor $\mathcal{C} \longrightarrow \mathcal{E}^{\mathrm{at}}$
whose image is dense, induces an equivalence

$$\mathcal{E} \xrightarrow{\sim} \widehat{\mathcal{C}}_J = \{\text{category of sheaves on } (\mathcal{C}, J)\}$$

if and only if J is the "atomic topology" on $\mathcal{C}$ for which
covering families are non-empty families of morphisms.

(iii) If $\mathcal{C}$ is an essentially small category,
the "atomic topology" J on $\mathcal{C}$ is well-defined if and only if any pair of morphisms

$$\begin{array}{ccc} & & \bullet \\ & & \downarrow \\ \bullet & \longrightarrow & \bullet \end{array}$$

can be completed to a commutative square:

$$\begin{array}{ccc} \bullet & \dashrightarrow & \bullet \\ \downarrow & & \downarrow \\ \bullet & \longrightarrow & \bullet \end{array}$$

In that case the topos $\widehat{\mathcal{C}}_J$ is atomic.
Furthermore, $\widehat{\mathcal{C}}_J$ is 2-valued if and only if $\mathcal{C}$ is connected.

Application of the duality of topos theory and logic:

Definition. –
A first-order geometric theory $\mathbb{T}$ is called "presheaf-type" if its classifying topos $\mathcal{E}_{\mathbb{T}}$ is equivalent to a topos of presheaves

$$\mathcal{E}_{\mathbb{T}} \xrightarrow{\sim} \widehat{\mathcal{C}} \quad \text{on some small category } \mathcal{C}.$$

Theorem (Caramello). –
Consider a "presheaf-type" theory $\mathbb{T}$ and a representation

$$\mathcal{E}_{\mathbb{T}} \xrightarrow{\sim} \widehat{\mathcal{C}}\,.$$

Then:

(i) *Any theory $\mathbb{T}'$ which is a "quotient" of $\mathbb{T}$ (in the sense that it has the same language and more axioms) defines a unique topology J on $\mathcal{C}$ such that*

$$\mathcal{E}_{\mathbb{T}'} \xrightarrow{\sim} \widehat{\mathcal{C}}_J\,.$$

(ii) *Conversely, for any topology J on $\mathcal{C}$, there is a "quotient" theory $\mathbb{T}'$ of $\mathbb{T}$ such that*

$$\mathcal{E}_{\mathbb{T}'} \xrightarrow{\sim} \widehat{\mathcal{C}}_J\,.$$

It is unique up to provable equivalence of families of axioms.

From syntax to semantics:

Proposition. – *Let $\mathbb{T}$ be a "presheaf-type" theory with*

$$\mathcal{E}_{\mathbb{T}} \xrightarrow{\sim} \widehat{\mathcal{C}}\,.$$

Then:

(i) *The category of set-valued models of $\mathbb{T}$ is equivalent to*

$$\mathrm{Ind}(\mathcal{C}^{\mathrm{op}})\,.$$

(ii) *The full subcategory $\mathcal{M}$ of set-valued models which are "finitely presentable" is equivalent to the Karoubi completion*

$$\mathcal{M} \xrightarrow{\sim} \mathrm{Kar}(\mathcal{C}^{\mathrm{op}})$$

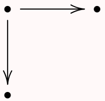
and there is also an equivalence

$$\mathcal{E}_{\mathbb{T}} \xrightarrow{\sim} \widehat{\mathcal{M}^{\mathrm{op}}}\,.$$

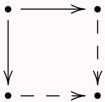
Proposition. –

In this situation, we have:

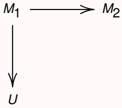
(i) *The atomic topology J is well-defined on $\mathcal{C}$ or $\mathcal{M}^{\mathrm{op}}$ if and only if any pair of morphisms of $\mathcal{M}$*



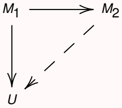
completes to a commutative square:



(ii) *In that case, a model U of $\mathbb{T}$ is a model of the quotient theory $\mathbb{T}'$ defined by J if and only if it is "homogeneous" in the sense that any diagram*



completes to a commutative triangle:



Categories of models:

• Start with an essentially small category $\mathcal{M}$ which is Karoubi-complete.
It can be presented as the category of finitely presentable set-valued models of some "presheaf-type" theory $\mathbb{T}$, with

$$\mathcal{E}_{\mathbb{T}} \xrightarrow{\sim} \widehat{\mathcal{M}^{\mathrm{op}}}.$$

Lemma. –
Suppose the atomic topology J is well-defined on a small category $\mathcal{C}$.
Then the canonical functor

$$\ell : \mathcal{C} \xrightarrow{y} \widehat{\mathcal{C}} \xrightarrow{j^*} \widehat{\mathcal{C}}_J$$

is fully faithful if and only if
any morphism of $\mathcal{C}^{\mathrm{op}}$ is a "strict" monomorphism,
i.e. a monomorphism $X \hookrightarrow Y$ whose image is defined by
equations $\quad u_i = v_i, \left(Y \underset{v_i}{\overset{u_i}{\rightrightarrows}} Y_i \right)_{i \in I}.$

Consequence. – It will be natural to start with a category of models

$$\mathcal{M}$$

whose morphisms are (strict) monomorphisms.
Such a category is necessarily Karoubi-complete.

Models and homogeneous models:

- Start with an essentially small category $\mathcal{M}$ such that
  - all morphisms of $\mathcal{M}$ are (strict) monomorphisms,
  - any pair of morphisms $\begin{array}{ccc} \bullet & \longrightarrow & \bullet \\ \downarrow & & \\ \bullet & & \end{array}$ completes to a commutative square: $\begin{array}{ccc} \bullet & \longrightarrow & \bullet \\ \downarrow & & \downarrow \\ \bullet & \longrightarrow & \bullet \end{array}$
- Write $\mathcal{M}$ as the category of finitely presentable set-valued models of some "presheaf-type" theory $\mathbb{T}$, with

$$\mathcal{E}_{\mathbb{T}} \xrightarrow{\sim} \widehat{\mathcal{M}^{\mathrm{op}}}.$$

- Consider the atomic topology J on $\mathcal{C} = \mathcal{M}^{\mathrm{op}}$ and the associated quotient theory $\mathbb{T}'$ of $\mathbb{T}$ (with $\mathcal{E}_{\mathbb{T}'} \xrightarrow{\sim} \widehat{\mathcal{C}}_J$) which is the theory of homogeneous models of $\mathbb{T}$.

Lemma. –

Suppose the topos $\mathcal{E}_{\mathbb{T}'} \cong \widehat{\mathcal{C}}_J$ *is 2-valued*
(i.e. the categories $\mathcal{C}$ *and* $\mathcal{M} = \mathcal{C}^{\mathrm{op}}$ *are connected).*
Then any homogeneous model U is also "universal"
in the sense that any object M of $\mathcal{M}$ *admits a morphism*

$$M \longrightarrow U\,.$$

Remark. –

Any such U is, in the category of set-valued models of $\mathbb{T}$,
a filtering colimit of finitely presentable models $M \in \mathcal{M}$.

Homogeneous fields:

- Start with the category $\mathcal{M}$ of finitely presented fields $K \supseteq \mathbb{Q}$.
  Its morphisms are strict monomorphisms.
- Present $\mathcal{M}$ as the category of finitely presentable set-valued models of some "presheaf-type" theory $\mathbb{T}$.

Lemma. – *For that purpose, it is enough to consider the usual axiomatization of the theory of fields of characteristic* 0 *and add a symbol (together with defining axioms) allowing to name the property of elements x of fields $K \supseteq \mathbb{Q}$:*

"x is not algebraic over $\mathbb{Q}$".

Proposition. –

(i) *The atomic topology J is well-defined on $\mathcal{C} = \mathcal{M}^{\mathrm{op}}$, and there is a quotient theory $\mathbb{T}'$ of $\mathbb{T}$ of "homogeneous fields", with*

$$\mathcal{E}_{\mathbb{T}'} \cong \widehat{\mathcal{C}}_J \quad .$$

(ii) *The topos $\mathcal{E}_{\mathbb{T}'}$ is 2-valued, so that $\mathbb{T}'$ is complete.*

(iii) *A field $K \supseteq \mathbb{Q}$ is homogeneous if an only if*

$$\begin{cases} - & \text{it is algebraically closed,} \\ - & \text{it has infinite transcendence degree over } \mathbb{Q}. \end{cases}$$

Our first ambition: compute the cohomology of a point!

- This is the case when the "geometric" category $\mathcal{G}$ consists in one object $\{\bullet\}$ and the identity morphism.
- In that case, cohomology functors should consist in only one vector space V over a coefficient field K. We wonder whether we can get the formula

$$\dim_K V = 1\,.$$

Definition. – *Let $\mathcal{M}$ be the category whose objects consist in*

$$\begin{cases} K = \text{field finitely presented over } \mathbb{Q}, \\ V = \text{finite dimensional vector space over } K, \end{cases}$$

and whose morphisms

$$(K, V) \longrightarrow (K', V')$$

consist in pairs

$$\begin{cases} (K \to K') = \text{morphism of fields}, \\ (V \to V') = K\text{-linear map} \end{cases}$$

such that

$$\text{not only } V \to V' \text{ is injective}$$

but, for any $k \geq 1$, the induced morphism

$$\mathrm{Sym}_K^k V \longrightarrow \mathrm{Sym}_{K'}^k V' \text{ is injective.}$$

Why the condition on the symmetric powers?

Lemma. – *Consider a pair of morphisms between objects of* $\mathcal{M}$

$$\begin{array}{ccc} (K,V) & \to & (K',V') \\ \downarrow & & \\ (K'',V'') & & \end{array}$$

where
$$\begin{cases} \bullet & K \to K' \text{ and } K \to K'' \text{ are morphisms of fields over } \mathbb{Q}, \\ \bullet & V \to V' \text{ and } V \to V'' \text{ are } K\text{-linear maps,} \\ \bullet & V' \text{ and } V'' \text{ have dimension } 1 \text{ over } K' \text{ and } K''. \end{cases}$$

Let $v_1, \cdots, v_d$ be a basis of V over K. Let v' and v'' be the images of v_1 in V' and V''. Supposing they are non-zero, write the images of $v_2, \cdots, v_d$

$$\begin{cases} \mu'_2 \cdot v', \cdots, \mu'_d \cdot v' & \text{with} \quad \mu'_i \in K', \ 2 \leq i \leq d, \\ \mu''_2 \cdot v'', \cdots, \mu''_d \cdot v'' & \text{with} \quad \mu''_i \in K'', \ 2 \leq i \leq d. \end{cases} \qquad \textit{Then:}$$

(i) *In order to be able to complete the diagram in a commutative square,*
we need $(\mu'_2, \cdots, \mu'_d) \in K'^{d-1}$ *and* $(\mu''_2, \cdots, \mu''_d) \in K''^{d-1}$
to verify exactly the same conditions of algebraic dependence over K.

(ii) *These conditions are not necessarily verified if we only suppose that $V \to V'$ and $V \to V''$ are injective.*

(iii) *If all $\mathrm{Sym}^k_K V \to \mathrm{Sym}^k_{K'} V'$ or $\mathrm{Sym}^k_K V \to \mathrm{Sym}^k_{K''} V''$ are injective, the elements $(\mu'_2, \cdots, \mu'_d)$ or $(\mu''_2, \cdots, \mu''_d)$ are algebraically independent over K.*

Monomorphisms which reduce the dimension to 1:

Lemma. –
Consider an object (K,V) of $\mathcal{M}$. Then there is a morphism of $\mathcal{M}$

$$(K,V) \longrightarrow (K',V')$$

such that

$$\dim_{K'} V' = 1\,.$$

Proof:
Let $v_1,\cdots,v_d$ be a basis of V over K.
Define

$$K' = K(X_1,\cdots,X_d)$$

where $X_1,\cdots,X_d$ are algebraically independent variables.
Let V' be a 1-dim vector space over K', with basis vector v'.
Supplement the embedding

$$K \hookrightarrow K(X_1,\cdots,X_d) = K'$$

with the K-linear map

$$\begin{array}{ccc} V & \longrightarrow & V' \\ v_i & \longmapsto & X_i \cdot v'\,, \quad 1 \le i \le d\,. \end{array}$$

This defines a morphism of $\mathcal{M}$

$$(K,V) \longrightarrow (K',V')\,.$$

□

Dimension of the cohomology space of a point:

Theorem (O. Caramello, L.L., Gonçalo Tabuada). –

(i) *The essentially small category $\mathcal{M}$ can be presented as the category of finitely presentable set-valued models of an explicit theory $\mathbb{T}$ of "presheaf-type", with* $\mathcal{E}_{\mathbb{T}} \cong \widehat{\mathcal{M}^{\rm op}}$.

(ii) *In the category $\mathcal{M}$, pairs of morphisms*

$$\begin{array}{ccc} \bullet & \longrightarrow & \bullet \\ \downarrow & & \\ \bullet & & \end{array}$$

can be completed to commutative squares

$$\begin{array}{ccc} \bullet & \longrightarrow & \bullet \\ \downarrow & & \vdots \\ \bullet & \dashrightarrow & \bullet \end{array}$$

so that the atomic topology J is well-defined on $\mathcal{C} = \mathcal{M}^{\rm op}$ and there exists a quotient theory $\mathbb{T}'$ of homogeneous $\mathbb{T}$-models, with

$$\mathcal{E}_{\mathbb{T}'} \cong \widehat{\mathcal{C}}_J \, .$$

(iii) *The category $\mathcal{M}$ has an initial object $(\mathbb{Q}, 0)$. A fortiori, it is connected, the topos $\widehat{\mathcal{C}}_J$ is 2-valued and the theory $\mathbb{T}'$ is complete.*

(iv) *A set-valued model (K, V) of $\mathbb{T}$ is homogeneous if and only if*

$$\begin{cases} - & \text{the field } K \supseteq \mathbb{Q} \text{ is homogeneous,} \\ - & \dim_K V = 1 \, . \end{cases}$$

Our second ambition: compute H^0 for Galois categories!

- This is the situation where the "geometric" category $\mathcal{G}$ is the category of finite sets endowed with a continuous action of some profinite group G.
- We wonder whether the general construction scheme

  $\begin{cases} \text{theory of presheaf type} \\ \to \text{induced quotient theory of homogeneous models} \end{cases}$

  allows to recover usual H^0 functors

  $$\begin{array}{rcccl} H^0 & : & \mathcal{G}^{\mathrm{op}} & \longrightarrow & K\text{-vect}, \\ & & S & \longmapsto & \{\text{space of maps } S \to K\} \end{array}$$

  with coefficients in
  homogeneous fields K
  (such as, in particular, $\mathbb{C}$ or $\overline{\mathbb{Q}}_\ell$).
- We will consider the case when
  $G =$ finite group.
  The general case reduces to this case.

Observations on structures and properties of H^0 functors:

Lemma. – *Consider a functor of the form*

$$\begin{array}{cccc} H^0 & : & \mathcal{G}^{\mathrm{op}} & \longrightarrow & K\text{-vect}, \\ & & S & \longmapsto & \{\text{space of maps } S \to K\}. \\ & & \| & & \\ & & \text{finite set} & & \\ & & \text{endowed with} & & \\ & & \text{an action of } G & & \end{array}$$

Then:

(i) *This presheaf of $\mathcal{G}$ is a sheaf for the étale topology of $\mathcal{G}$ (for which a family $(S_i \to S)_{i \in I}$ is a covering if it is globally surjective).*

(ii) *There is a functorial family of K-linear maps*

$$H^0(S) \otimes_K H^0(S') \longrightarrow H^0(S \times S')$$

or, equivalently,

$$H^0 : \mathcal{G}^{\mathrm{op}} \longrightarrow K\text{-vect}$$

is a presheaf of (commutative) K-algebras.

(iii) *Each commutative K-algebra $H^0(S)$ has no nilpotents, and each structure morphism*

$$H^0(S) \otimes_K H^0(S') \longrightarrow H^0(S \times S')$$

is an isomorphism.

Consequences for defining a starting presheaf-type theory:

- We want to define a presheaf-type theory $\mathbb{T}$

whose finitely presentable models can appear as subobjects of cohomology functors H^0.

This means that they should inherit all structures and properties of H^0 functors which are inherited by arbitrary subobjects.

Definition. – *Let $\mathcal{M}$ be the category defined in the following way:*

(i) *Objects are pairs $(K, F : \mathcal{G}^{\mathrm{op}} \to K\text{-alg})$ where*

$$\begin{cases} \bullet \quad K \text{ is a finitely presented field over } \mathbb{Q}, \\ \bullet \quad F : \mathcal{G}^{\mathrm{op}} \to K\text{-alg is a presheaf of } K\text{-algebras} \end{cases}$$

such that

- *the presheaf F is separated for the étale topology of $\mathcal{G}$,*
- *each K-algebra $F(S)$ is commutative, and has no nilpotents,*
- *the presheaf of K-algebras F is generated by finitely many elements,*
- *each induced morphism $F(S) \otimes_K F(S') \longrightarrow F(S \times S')$ is injective.*

(ii) *Morphisms $(K, F) \to (K', F')$ consist in*

$$\begin{cases} \bullet \quad \text{a morphism } K \to K' \text{ of fields,} \\ \bullet \quad \text{a morphism } F \to F' \text{ of presheaves of } K\text{-algebras} \\ \quad\;\; \text{such that each component } F(S) \to F'(S) \text{ is injective.} \end{cases}$$

Reduction to sheaves:

Lemma. – *Let $(K, F : \mathcal{G}^{\text{op}} \to K\text{-alg})$ be an object of $\mathcal{M}$.*
Let $\widetilde{F}$ be the transform of F by the sheafification functor relatively to the étale topology of $\mathcal{G}$. Then:

(i) *The sheaf $\widetilde{F}$ inherits from F all its structures.*

(ii) *The pair $(K, \widetilde{F} : \mathcal{G}^{\text{op}} \to K\text{-alg})$ has all properties defining objects of $\mathcal{M}$, except for finite generation.*
It is a filtering colimit of objects of $\mathcal{M}$, i.e. an object of $\text{Ind}(\mathcal{M})$.

(iii) *The canonical morphism of presheaves $F \longrightarrow \widetilde{F}$ respects all structures and has injective components $F(S) \longrightarrow \widetilde{F}(S)$.*
It is a morphism of $\text{Ind}(\mathcal{M})$.

Proof:

(i) This comes from the fact that the sheafification functor transforms linear presheaves into linear sheaves and respects tensor products.

(ii) The sheaf $\widetilde{F}$ is a fortiori a separated presheaf. Any nilpotent element of some $\widetilde{F}(S)$ is 0 because it is 0 locally on S. The maps $\widetilde{F}(S) \otimes_K \widetilde{F}(S') \to \widetilde{F}(S \times S')$ are injective because sheafification respects monomorphisms of presheaves.

(iii) The components $F(S) \to \widetilde{F}(S)$ are injective because F is separated. □

Reduction of sheaves to split algebras:

• Let S_u be the "universal" object of $\mathcal{G}$ consisting in the set G endowed with the action of G by translation.

Lemma. – *Let $(K, F : \mathcal{G}^{\mathrm{op}} \to K\text{-alg})$ be an object of $\mathcal{M}$. Let $\widetilde{F}$ be the sheafification of F. Then:*

(i) *The sheaf $\widetilde{F}$ is entirely determined by the K-algebra* $\quad A_F = \widetilde{F}(S_u)$.

(ii) *One has* $\quad A_F = \widetilde{F}(S_u) = F(S_u)$.

(iii) *The dimension of A_F on K is at most $|G| = |S_u|$.*

(iv) *One can choose a finite extension K' of K so that the algebra*

$$A_F \otimes_K K'$$

is split, i.e. isomorphic to a product $\quad \prod_{p \in P_F} K'$

where P_F is a set endowed with an action of G.

(v) *The action of G on P_F is transitive.*

(vi) *The pair $(K', F \otimes_K K')$ is an object of $\mathcal{M}$, the morphism $(K, F) \to (K', F \otimes_K K')$ is a morphism of $\mathcal{M}$ and we have $\widetilde{F \otimes_K K'} = \widetilde{F} \otimes_K K'$.*

Verification of the reduction to split algebras:

Proof of the lemma:

- **(i)** The sheaf $\widetilde{F}$ is entirely determined by $\widetilde{F}(S_u) = A_F$ because any object S of $\mathcal{M}$ is covered by copies of S_u.
- **(ii)** One has $\widetilde{F}(S_u) = F(S_u)$ because the object S_u of $\mathcal{G}$ has no non trivial cover for the étale topology.
- **(iii)** The decomposition $S_u \times S_u = \coprod_{g \in G} S_u$ applied to the injectivity of the map $F(S_u) \otimes F(S_u) \longrightarrow F(S_u \times S_u)$ implies that the morphism of representations of $G \times G$
$$A_F \otimes_K A_F \longrightarrow \mathrm{Ind}_G^{G \times G}(A_F)$$
is a monomorphism.
This implies that $\dim_K A_F \leq |G|$.
- **(iv)** The commutative algebra A_F is finite-dimensional and has no nilpotents. So it can be split by a finite extension K' of K.
- **(v)** If $A_F \otimes_K K' = \prod_{p \in P_F} K'$, the action of G on P_F is transitive because the morphism $A_F \otimes_K A_F \longrightarrow \mathrm{Ind}_G^{G \times G}(A_F)$ is injective.
- **(vi)** comes from the fact that the sheafification functor respects tensor products and monomorphisms.

□

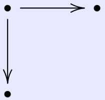
Computation of H^0 functors of a Galois category:

The previous lemma suffices to prove:

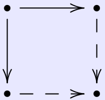
Theorem (O. Caramello, L.L., Gonçalo Tabuada). –

(i) *The essentially small category $\mathcal{M}$ can be presented as the category of finitely presentable set-valued models of an explicit theory $\mathbb{T}$ of "presheaf-type", with* $\mathcal{E}_{\mathbb{T}} \cong \widehat{\mathcal{M}^{\mathrm{op}}}$.

(ii) *In the category $\mathcal{M}$, pairs of morphisms*



can be completed to commutative squares



so that the atomic topology J is well-defined on $\mathcal{C} = \mathcal{M}^{\mathrm{op}}$ and there exists a quotient theory $\mathbb{T}'$ of homogeneous $\mathbb{T}$-models, with

$$\mathcal{E}_{\mathbb{T}'} \cong \widehat{\mathcal{C}}_J .$$

(iii) *The category $\mathcal{M}$ has an initial object $(\mathbb{Q}, 0)$. A fortiori, it is connected, the topos $\widehat{\mathcal{C}}_J$ is 2-valued and the theory $\mathbb{T}'$ is complete.*

(iv) *A set-valued model (K, F) of $\mathbb{T}$ is homogeneous if and only if*

- *the field $K \supseteq \mathbb{Q}$ is homogeneous,*
- *F is the "regular" sheaf on $\mathcal{G}$* $\quad S \mapsto F(S) = \{$*space of maps* $S \to K\}$.