Grothendieck toposes as bridges between geometry, meaning and formal languages

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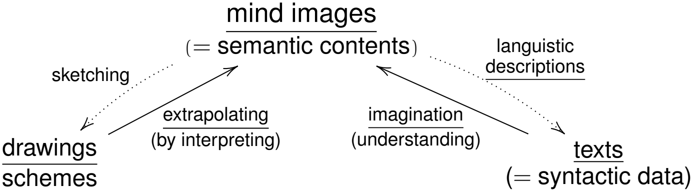
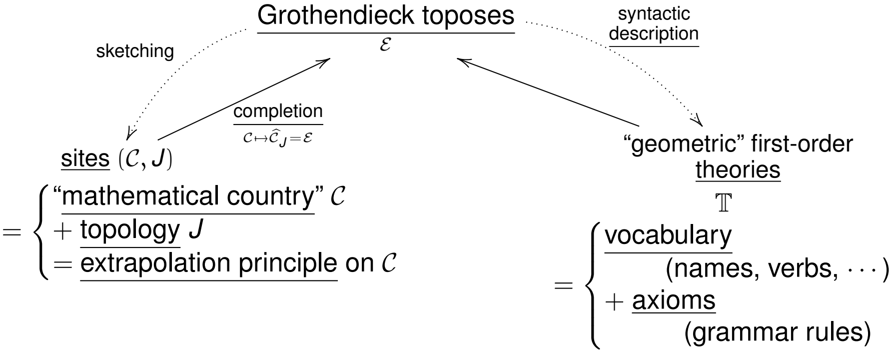
The double expression of semantic contents and their modelling by Grothendieck topos theory:



Proposed mathematical modelling:



Mathematical countries:

Definition. – *A "mathematical country" (or "category") consists in*

- *cities* $A, B, \cdots$,
- *itineraries* $A \to B$ *between cities,*
- *an associative composition law of itineraries*

$$\left(A \xrightarrow{f} B \xrightarrow{g} C\right) \longmapsto \left(A \xrightarrow{g\circ f} C\right) \qquad \text{which admits "units"} \quad A \xrightarrow{\mathrm{id}_A} A.$$

Examples of mathematical countries:

- the country of groups and group homomorphisms,
- the country of topological spaces and continuous maps,
- for any group G, the country consisting in
  - — a unique city denoted G,
  - — itineraries $G \to G$ which are the elements g of G,
  - — the composition law of elements of G,
- for any topological space X, the country consisting in
  - — cities which are the open subsets $U \subseteq X$,
  - — itineraries $U \to V$ which are the inclusion relations $U \subseteq V$,
  - — the composition of inclusion relations.

The country of mathematical countries:

Definition. – *A mathematical country $\mathcal{C}$ is called*

- *"locally small" if, for any cities A, B of $\mathcal{C}$, the itineraries $A \to B$ make up a set $\mathrm{Hom}(A, B)$,*
- *"small" if, furthermore, the cities of $\mathcal{C}$ make up a set.*

Definition. –
An (international) "twinning" (or "functor") between two mathematical countries $F : \mathcal{C} \longrightarrow \mathcal{D}$ consists in associating

$$\begin{cases} \bullet \text{ with any city } X \text{ of } \mathcal{C} \text{ a city } F(X) \text{ of } \mathcal{D}, \\ \bullet \text{ with any itinerary } X \xrightarrow{f} Y \text{ of } \mathcal{C} \text{ an itinerary } F(X) \xrightarrow{F(f)} F(Y) \text{ of } \mathcal{D}, \end{cases}$$

so as to respect compositions $X \xrightarrow{f} Y \xrightarrow{g} Z$ in the sense that $F(g \circ f) = F(g) \circ F(f)$.

Observations :

- Twinnings naturally compose

$$\left(\mathcal{C} \xrightarrow{F} \mathcal{C}' \xrightarrow{G} \mathcal{C}''\right) \longmapsto \left(\mathcal{C} \xrightarrow{G \circ F} \mathcal{C}''\right).$$

- This defines a country where $\begin{cases} - & \text{the cities are small countries,} \\ - & \text{the itineraries are twinnings.} \end{cases}$

Countries of twinnings:

Definition. – *If* $\mathcal{C} \overset{F}{\underset{G}{\rightrightarrows}} \mathcal{D}$ *are a couple of twinnings between two countries,* *a "passage" between these twinnings (or "functor transform")* $\rho : F \to G$ *consists in associating with any city* X *of* $\mathcal{C}$

an itinerary $\rho(X) : F(X) \to G(X)$ *of* $\mathcal{D}$*,*

so that, for any itinerary $X \xrightarrow{f} Y$ *of* $\mathcal{C}$*, there is a "commutative square"*

$$\begin{array}{ccc} F(X) & \xrightarrow{F(f)} & F(Y) \\ {\scriptstyle \rho(X)}\downarrow & & \downarrow{\scriptstyle \rho(Y)} \\ G(X) & \xrightarrow{G(f)} & G(Y) \end{array}$$

in the sense that $\rho(Y) \circ F(f) = G(f) \circ \rho(X)$ *in* $\mathcal{D}$*.*

Observations :

- Passages between twinnings from $\mathcal{C}$ to $\mathcal{D}$ naturally compose

$$(F \xrightarrow{\rho} G \xrightarrow{\rho'} H) \longmapsto (F \xrightarrow{\rho' \circ \rho} H)\,.$$

- This defines a country $[\mathcal{C}, \mathcal{D}]$ where
  - – the cities are the twinnings $\mathcal{C} \to \mathcal{D}$,
  - – the itineraries are the passages between twinnings.
- If $\mathcal{C}$ is a small country and $\mathcal{D}$ is locally small, $[\mathcal{C}, \mathcal{D}]$ is locally small.

The reflection of a city in itineraries leading to this city:

Definition. – *If $\mathcal{C}$ is a locally small mathematical country, the reflection of a city X of $\mathcal{C}$ is the double map which associates*

- *with any city A of $\mathcal{C}$ the set*
$$\mathrm{Hom}(A,X) = \{\textit{itineraries } A \to X \textit{ of } \mathcal{C}\},$$
- *with any itinerary $A \to B$ of $\mathcal{C}$ the composition application*
$$\begin{array}{ccc} \mathrm{Hom}(B,X) & \xrightarrow{\bullet\circ f} & \mathrm{Hom}(A,X)\,, \\ (B \xrightarrow{b} X) & \longmapsto & (A \xrightarrow{b\circ f} X)\,. \end{array}$$

Lemma. –

(i) *For any city X of $\mathcal{C}$, $y(X)$ is a city of the country*
$$\widehat{\mathcal{C}} = [\mathcal{C}^{\mathrm{op}}, \mathrm{Set}] = \textit{country of twinnings } \mathcal{C}^{\mathrm{op}} \to \mathrm{Set},$$
where
$$\mathcal{C}^{\mathrm{op}} = \textit{country whose} \begin{cases} \textit{cities are the cities of } \mathcal{C}, \\ \textit{itineraries } B \to A \textit{ are the itineraries } A \to B \textit{ of } \mathcal{C}. \end{cases}$$

(ii) *Any itinerary $X \xrightarrow{f} Y$ of $\mathcal{C}$ defines a passage*
$$y(f) : y(X) \longrightarrow y(Y)\,.$$

Looking at a country through its reflection:

Lemma (Yoneda). –

(i) *Associating* $\begin{cases} - & \text{with any city } X \text{ of } \mathcal{C} \text{ its reflection } y(X), \\ - & \text{with any itinerary } X \xrightarrow{f} Y \text{ of } \mathcal{C} \text{ the passage} \\ & \quad y(f) : y(X) \longrightarrow y(Y) \end{cases}$

defines a twinning

$$y : \mathcal{C} \longrightarrow \widehat{\mathcal{C}} = [\mathcal{C}^{\mathrm{op}}, \mathrm{Set}] \,.$$

(ii) *This twinning $y : \mathcal{C} \longrightarrow \widehat{\mathcal{C}}$ is "fully faithful" in the sense that, for any cities X, Y of $\mathcal{C}$, the map*

$$\begin{array}{ccc} \mathrm{Hom}(X, Y) & \longrightarrow & \mathrm{Hom}(y(X), y(Y)) \,, \\ (X \xrightarrow{f} Y) & \longmapsto & (y(X) \xrightarrow{y(f)} y(Y)) \end{array}$$

is one-to-one.

Consequences :

- Any city X of $\mathcal{C}$ is characterized (up to invertible itinerary) by its reflection $y(X)$ in $\widehat{\mathcal{C}}$.
- A city P of $\widehat{\mathcal{C}}$ (or "potential city" of $\mathcal{C}$) is called "representable" (or "real") if there exists a city X of $\mathcal{C}$ such that $y(X) \cong P$.

The extraordinary properties of countries of reflections:

Proposition. – *For any "mathematical country" $\mathcal{C}$ which is small, the country $\widehat{\mathcal{C}}$ of its "reflections" (or "presheaves") has the same constructive properties as the country* Set *of sets:*

(0) *It is locally small: itineraries between pairs of cities make up sets.*

(1) *Finite and infinite products* $\prod_{i \in I} P_i$ *of cities are always well-defined as well as "fiber products"*

$$S' \times_S P \text{ defined by equations } s = p \text{ in: } \begin{array}{ccc} & & P \\ & & \downarrow p \\ S' & \xrightarrow{s} & S \end{array}$$

(2) *Finite and infinite sums* $\coprod_{i \in I} P_i$ *are well-defined and disjoint, and relations* $R \rightrightarrows P$ *always define quotients* $P \twoheadrightarrow P'$.

(3) *Fiber products* $S' \times_S \bullet$ *over any itinerary* $S' \to S$ *respect arbitrary sums and quotients by relations.*

(4) *For any city* P, *its quotients* $P \twoheadrightarrow P'$ *correspond one-to-one to equivalence relations* $R \hookrightarrow P \times P$, *in such a way that* $R = P \times_{P'} P$.

Completions of "mathematical countries" :

Definition. – *Let $\mathcal{C}$ be a "mathematical country" which is small.*
Let's call "completion" of $\mathcal{C}$
any twinning with a "completed mathematical country" $\mathcal{E}$

$$\ell : \mathcal{C} \longrightarrow \mathcal{E}$$

such that

- *$\mathcal{E}$ has the same properties (0), (1), (2), (3), (4) as the country of sets,*
- *for any cities E_1, E_2 of $\mathcal{E}$, itineraries $E_1 \to E_2$ correspond one-to-one to families of maps*

  $$\mathrm{Hom}(\ell(X), E_1) \longrightarrow \mathrm{Hom}(\ell(X), E_2)$$

  indexed by cities X of $\mathcal{C}$, which are compatible in the sense that, for any itinerary $X \to Y$ of $\mathcal{C}$,

  the square
  $$\begin{array}{ccc} \mathrm{Hom}(\ell(Y), E_1) & \longrightarrow & \mathrm{Hom}(\ell(Y), E_2) \\ \downarrow & & \downarrow \\ \mathrm{Hom}(\ell(X), E_1) & \longrightarrow & \mathrm{Hom}(\ell(X), E_2) \end{array}$$
  commutes.

Completions and notions of coverings:

Definition. –
Consider a completion $\ell : \mathcal{C} \to \mathcal{E}$ *of a small "mathematical country"* $\mathcal{C}$.
We say that a family of itineraries of $\mathcal{C}$ *leading to a city* X

$$X_i \xrightarrow{x_i} X\,, \quad i \in I\,,$$

is a covering of X *if, in the completion* $\mathcal{E}$,
the itineraries $\ell(x_i) : \ell(X_i) \to \ell(X)$ *make* $\ell(X)$ *appear as a quotient of* $\coprod_{i\in I} \ell(X_i)$.

Lemma. – *The notion of covering defined by a completion* $\ell : \mathcal{C} \to \mathcal{E}$ *has the following following properties:*

(A) Any unit itinerary $X \xrightarrow{\mathrm{id}_X} X$ *is a covering.*

(B) If $(X_i \xrightarrow{x_i} X)_{i\in I}$ *is a covering,*
then for any itinerary $X' \to X$ *there exists a covering* $(X_j' \xrightarrow{x_j'} X')$
such that all composites $X_j' \xrightarrow{x_j'} X' \to X$ *factorize through some* $X_i \xrightarrow{x_i} X$.

(C) If $(X_i \xrightarrow{x_i} X)_{i\in I}$ *is a covering and each* X_i *has a covering* $(X_{i,j} \xrightarrow{x_{i,j}} X_i)_{j\in I_i}$,
then the composites $X_{i,j} \xrightarrow{x_{i,j}} X_i \xrightarrow{x_i} X$ *make up a covering of* X.

(D) Any family $(X_i \xrightarrow{x_i} X)_{i\in I}$ *which contains a covering is a covering.*

Grothendieck topologies and coverings:

Definition. –
Let $\mathcal{C}$ be a small "mathematical country".
A "Grothendieck topology" on $\mathcal{C}$
is a notion of covering J
which respects conditions (A), (B), (C), (D) of the previous lemma.

Theorem. –

(i) *Any "completion" $\ell : \mathcal{C} \to \mathcal{E}$ of $\mathcal{C}$*
is characterized by the topology J it defines.

(ii) *Conversely, any topology J of $\mathcal{C}$*
defines a unique "completion"

$$\mathcal{C} \longrightarrow \widehat{\mathcal{C}}_J \, .$$

Remark:
A topology J of $\mathcal{C}$ can also be seen as an "extrapolation principle".
Indeed, it allows to extrapolate from $\mathcal{C}$
the components of the completion $\widehat{\mathcal{C}}_J$.

Grothendieck's sites and toposes:

Definition. – *A "site" is a pair $(\mathcal{C}, J)$ consisting in*

- *a small "mathematical country" $\mathcal{C}$,*
- *a topology J on $\mathcal{C}$,*
  i.e. a notion of covering of cities of $\mathcal{C}$ by families of itineraries.

Definition. – *A "topos" is a "mathematical country" $\mathcal{E}$ which can be constructed as a completion*

$$\mathcal{E} \cong \widehat{\mathcal{C}}_J$$

of some sites $(\mathcal{C}, J)$.

Remark:
Any topos has infinitely many different presentations

$$\mathcal{E} \cong \widehat{\mathcal{C}}_J\,.$$

For any such presentation,
$\mathcal{C}$ appears as a "sketch" of $\mathcal{E}$,
which allows to fully reconstruct $\mathcal{E}$
if it is completed with an "extrapolation principle" J.

The site and the topos of a topological space:

Definition. – *Let X be a topological space.*

(i) *It defines a site $(\mathcal{C}_X, J_X)$ consisting in*

- *a mathematical country $\mathcal{C}_X$ whose cities are open subsets $U \subseteq X$ and whose itineraries are inclusion relations $U' \hookrightarrow U$,*
- *a topology J_X on $\mathcal{C}_X$ for which coverings are families*

$$(U_i \hookrightarrow U)_{i \in I} \quad \text{such that} \quad U = \bigcup_{i \in I} U_i .$$

(ii) *This site defines a topos $\mathcal{E}_X$.*

Proposition. –
Let $f : X \to Y$ be a continuous map between topological spaces.
Then:

(i) *The formation of pull-backs of open subsets of Y by f defines a twinning* $\quad f^{-1} : \mathcal{C}_Y \longrightarrow \mathcal{C}_X$.

(ii) *This twinning extends to a twinning of completions* $\quad f^* : \mathcal{E}_Y \longrightarrow \mathcal{E}_X$ *which respects*

- *arbitrary sums and quotients by relations $R \rightrightarrows E$,*
- *finite products and fiber products $E_1 \times_E E_2$.*

The country of toposes:

Definition. – *An itinerary between two toposes*

$$f : \mathcal{E}' \longrightarrow \mathcal{E}$$

is defined as a twinning in the reverse direction

$$f^* : \mathcal{E} \longrightarrow \mathcal{E}'$$

which respects

- *arbitrary sums and quotients by relations,*
- *finite products ands fiber products.*

Theorem. – *If X, Y are topological spaces and Y is "sober", continuous maps*

$$f : X \longrightarrow Y$$

correspond one-to-one to itineraries of toposes

$$\mathcal{E}_X \longrightarrow \mathcal{E}_Y \, .$$

Remark: In particular, points of Y correspond one-to-one to topos itineraries

$$\mathrm{Set} \longrightarrow \mathcal{E}_Y \, .$$

For that reason, "points" of a topos $\mathcal{E}$
are defined as topos itineraries $\quad \mathrm{Set} \longrightarrow \mathcal{E} \, .$

Geometry of toposes:

All usual notions of topology generalize in the context of toposes, in particular the notions of submersion and immersion:

Definition. –

(i) *An itinerary of toposes $f : \mathcal{E}' \to \mathcal{E}$ is a "submersion" $\mathcal{E}' \twoheadrightarrow \mathcal{E}$ if, for any cities E_1, E_2 of $\mathcal{E}$, the map*

$$f^* : \mathrm{Hom}(E_1, E_2) \longrightarrow \mathrm{Hom}(f^* E_1, f^* E_2)$$

is injective.

(ii) *An itinerary of toposes $f : \mathcal{E}' \to \mathcal{E}$ is an "immersion" (so that $\mathcal{E}'$ becomes a "subtopos" of $\mathcal{E}$) if, for any cities E_1, E_2 of $\mathcal{E}'$, itineraries $E_1 \to E_2$ of $\mathcal{E}'$ correspond one-to-one to families of maps indexed by cities E of $\mathcal{E}$*

$$\mathrm{Hom}(f^* E, E_1) \to \mathrm{Hom}(f^* E, E_2)$$

which are compatible in the sense that, for any itinerary $E' \to E$ of $\mathcal{E}$, the square

$$\begin{array}{ccc} \mathrm{Hom}(f^* E, E_1) & \longrightarrow & \mathrm{Hom}(f^* E, E_2) \\ \downarrow & & \downarrow \\ \mathrm{Hom}(f^* E', E_1) & \longrightarrow & \mathrm{Hom}(f^* E', E_2) \end{array}$$

commutes.

Geometry of subtoposes:

Proposition. –

(i) *Subtoposes $\mathcal{E}' \hookrightarrow \mathcal{E}$ of a topos $\mathcal{E}$ make up an ordered set.*

(ii) *Any family of subtoposes $\mathcal{E}_i \hookrightarrow \mathcal{E}$, $i \in I$, has a join $\bigcup_{i\in I} \mathcal{E}_i \hookrightarrow \mathcal{E}$ and an intersection $\bigcap_{i\in I} \mathcal{E}_i \hookrightarrow \mathcal{E}$.*

(iii) *One always has $\mathcal{E}' \cup \big(\bigcap_{i\in I} \mathcal{E}_i\big) = \bigcap_{i\in I} (\mathcal{E}' \cup \mathcal{E}_i)$.*

Theorem. – *Any itinerary of toposes $\mathcal{E}' \xrightarrow{f} \mathcal{E}$ uniquely factorizes as*

$$\mathcal{E}' \twoheadrightarrow \mathrm{Im}(f) \hookrightarrow \mathcal{E}\,.$$

Theorem. –

(i) *Any itinerary of toposes $\mathcal{E}' \xrightarrow{f} \mathcal{E}$ defines an "image" map*

$$f_* : (\mathcal{E}'_1 \hookrightarrow \mathcal{E}') \longmapsto (\mathrm{Im}(\mathcal{E}'_1) \hookrightarrow \mathcal{E})\,.$$

This maps respects the ordering and arbitrary unions.

(ii) *It also defines a "pull-back" map* $f^{-1} : (\mathcal{E}_1 \hookrightarrow \mathcal{E}) \longmapsto (f^{-1}\mathcal{E}_1 \hookrightarrow \mathcal{E}')$ *characterized by the property that* $\mathcal{E}'_1 \subseteq f^{-1}\mathcal{E}_1 \Leftrightarrow f_*\mathcal{E}'_1 \subseteq \mathcal{E}_1\,.$ *The "pull-back" map respects the ordering, arbitrary intersections and finite unions.*

Syntax of formal languages or “theories”:

Definition. – *A “geometric” first-order theory* $\mathbb{T}$ *consists in:*

(1) a vocabulary Σ *comprising*

- *a family of “city names”*
  [such as: G (group), R (ring), M (module), $\cdots$ *],*
- *a family of “itineraries names”* $E_1 \cdots E_n \xrightarrow{e} E$
  [such as: $GG \xrightarrow{\cdot} G$*,* $G \xrightarrow{(\bullet)^{-1}} G$
  or $RR \xrightarrow{+} R$*,* $RR \xrightarrow{\cdot} R$*,* $R \xrightarrow{-(\bullet)} R, \cdots$ *],*
- *a family of “relation names”* $R \rightarrowtail E_1 \cdots E_n$
  [such as: $\leq \rightarrowtail EE, \sim \mapsto EE, \cdots$ *],*

(2) a list of axioms which have the form of implications $\varphi(\vec{x}) \vdash \psi(\vec{x})$ *where*

- $\vec{x} = (x_1^{E_1}, \cdots, x_n^{E_n})$ *is a finite family of variables* $x_i^{E_i}$
  associated with “city names” E_i*,*
- φ, ψ *are “formulas” in these variables*
  which are constructed from itineraries names or relation names of Σ
  and which interpret only in terms
  of images of itineraries, arbitrary unions and finite intersections.

Semantic expressions of formal languages:

Definition. – *Let $\mathbb{T}$ be a (geometric first-order) theory.*
Its "semantic expression" in a topos $\mathcal{E}$ is the mathematical country

$$\mathbb{T}\text{-mod}(\mathcal{E})$$

of "models" of $\mathbb{T}$ in $\mathcal{E}$ in the following sense:

(i) *The cities are "models" M consisting in*

- *cities ME of $\mathcal{E}$ indexed by city names E of $\mathbb{T}$,*
- *itineraries $ME_1 \times \cdots \times ME_n \xrightarrow{Me} ME$ of $\mathcal{E}$ indexed by itinerary names $e : E_1 \cdots E_n \to E$ of $\mathbb{T}$,*
- *immersions $MR \hookrightarrow ME_1 \times \cdots \times ME_n$ indexed by relation names $R \rightarrowtail E_1 \cdots E_n$ of $\mathbb{T}$,*

such that, for any axiom of $\mathbb{T}$

$$\varphi(\vec{x}) \vdash \psi(\vec{x}) \quad \text{in variables} \quad \vec{x} = (x_1^{E_1}, \cdots, x_n^{E_n}),$$

the corresponding interpretation immersions

$$M\varphi \longrightarrow ME_1 \times \cdots \times ME_n, \quad M\psi \longrightarrow ME_1 \times \cdots \times ME_n$$

are related by an inclusion

$$M\varphi \subseteq M\psi.$$

(ii) Itineraries of models of $\mathbb{T}$ in $\mathcal{E}$

$$M' \longrightarrow M$$

consist in families of itineraries of $\mathcal{E}$

$$M'E \longrightarrow ME \quad \text{indexed by “city names” } E \text{ of } \mathbb{T}$$

such that the following squares indexed by
itinerary names $e : E_1 \cdots E_n \to E$ and relation names $R \rightarrowtail E_1 \cdots E_n$

$$\begin{array}{ccc} M'E_1 \times \cdots \times M'E_n & \xrightarrow{M'e} & M'E \\ \downarrow & & \downarrow \\ ME_1 \times \cdots \times ME_n & \xrightarrow{Me} & ME \end{array}$$

$$\begin{array}{ccc} M'R & \hookrightarrow & M'E_1 \times \cdots \times M'E_n \\ \downarrow & & \downarrow \\ MR & \hookrightarrow & ME_1 \times \cdots \times ME_n \end{array}$$

commute.

The network of semantic expressions of a formal language:

Lemma. – *Let $\mathbb{T}$ be a (geometric first-order) theory.*
Any itinerary of toposes $f : \mathcal{E}' \longrightarrow \mathcal{E}$
naturally defines a twinning
between the semantic expressions of $\mathbb{T}$ in $\mathcal{E}$ and $\mathcal{E}'$

$$f^* : \mathbb{T}\text{-mod}(\mathcal{E}) \longrightarrow \mathbb{T}\text{-mod}(\mathcal{E}')\,.$$

Explanation : An itinerary $f : \mathcal{E}' \to \mathcal{E}$ consists by definition in a twinning

$$f^* : \mathcal{E} \longrightarrow \mathcal{E}'$$

which respects arbitrary sums, quotients, finite products and fiber products.
It transforms

- cities ME of $\mathcal{E}$ into cities f^*ME of $\mathcal{E}'$,
- itineraries $ME_1 \times \cdots \times ME_n \xrightarrow{Me} ME$ of $\mathcal{E}$
  into itineraries $f^*ME_1 \times \cdots \times f^*ME_n \xrightarrow{f^*Me} f^*ME$ of $\mathcal{E}'$,
- immersions $MR \hookrightarrow ME_1 \times \cdots \times ME_n$ of $\mathcal{E}$
  into immersions $f^*MR \hookrightarrow f^*ME_1 \times \cdots \times f^*ME_n$ of $\mathcal{E}'$.

Furthermore, it respects images, arbitrary unions and finite intersections, and, as a consequence, interpretations of formulas $\varphi(\vec{x})$, $\psi(\vec{x})$ which make up the axioms $\varphi(\vec{x}) \vdash \psi(\vec{x})$ of $\mathbb{T}$.

The network of countries of topos itineraries:

Definition. – *For any toposes $\mathcal{E}, \mathcal{E}'$, let's denote*

$$\mathrm{Geom}(\mathcal{E}', \mathcal{E})$$

the "mathematical country" defined as follows:

(i) *Its cities are topos itineraries*

$$f : \mathcal{E}' \longrightarrow \mathcal{E}$$

i.e. twinnings

$$f^* : \mathcal{E} \longrightarrow \mathcal{E}'$$

which respect sums, quotients, finite products and fiber products.

(ii) *Its itineraries*

$$(\mathcal{E}' \xrightarrow{f_1} \mathcal{E}) \xrightarrow{\rho} (\mathcal{E}' \xrightarrow{f_2} \mathcal{E})$$

are passages between the corresponding twinnings

$$\rho : f_1^* \longrightarrow f_2^* \, .$$

Lemma. –
Composition with any topos itinerary $\mathcal{E}_2' \xrightarrow{g} \mathcal{E}_1'$
defines a twinning between
countries of topos itineraries

$$\mathrm{Geom}(\mathcal{E}_1', \mathcal{E}) \longrightarrow \mathrm{Geom}(\mathcal{E}_2', \mathcal{E}) \, .$$

Theories of points of a topos :

If $\mathcal{E}$ is a topos, the country of points of $\mathcal{E}$ is by definition $\mathrm{pt}(\mathcal{E}) = \mathrm{Geom}(\mathrm{Set}, \mathcal{E})$.
More generally, any $\mathrm{Geom}(\mathcal{E}', \mathcal{E})$ can be called
the country of "$\mathcal{E}'$-parametrized points of $\mathcal{E}$".

Theorem. – *For any presentation of a topos $\mathcal{E}$ by a site $(\mathcal{C}, J)$*

$$\mathcal{E} \cong \widehat{\mathcal{C}}_J \,,$$

there exists a (geometric first-order) theory $\mathbb{T}_{\mathcal{C},J}$ such that

- *"city names" of $\mathbb{T}_{\mathcal{C},J}$ are cities X of $\mathcal{C}$,*
- *"itineraries names" of $\mathbb{T}_{\mathcal{C},J}$ are itineraries $X \to Y$ of $\mathcal{C}$,*
- *$\mathbb{T}_{\mathcal{C},J}$ has no "relation names",*

and which describes the points of $\mathcal{E}$ in the following sense:

(1) *Any topos $\mathcal{E}'$ defines an equivalence* $\mathrm{Geom}(\mathcal{E}', \mathcal{E}) \xrightarrow{\sim} \mathbb{T}_{\mathcal{C},J}\text{-}\mathrm{mod}(\mathcal{E}')\,.$

(2) *These equivalences are natural in the sense that,*

for any topos itinerary $\mathcal{E}'_2 \xrightarrow{f} \mathcal{E}'_1$, the square

$$\begin{array}{ccc} \mathrm{Geom}(\mathcal{E}'_1, \mathcal{E}) & \xrightarrow{\sim} & \mathbb{T}_{\mathcal{C},J}\text{-}\mathrm{mod}(\mathcal{E}'_1) \\ {\scriptstyle (\bullet)\circ f}\downarrow & & \downarrow{\scriptstyle f^*} \\ \mathrm{Geom}(\mathcal{E}'_2, \mathcal{E}) & \xrightarrow{\sim} & \mathbb{T}_{\mathcal{C},J}\text{-}\mathrm{mod}(\mathcal{E}'_2) \end{array}$$

is commutative.

The topos incarnation of the semantics of a formal language:

Theorem. – *Let $\mathbb{T}$ be a (geometric first-oder) theory.*
Then there exists a topos $\mathcal{E}_{\mathbb{T}}$ endowed with a model

$$U_{\mathbb{T}} \quad \textit{of} \quad \mathbb{T} \quad \textit{in} \quad \mathcal{E}_{\mathbb{T}},$$

such that, for any topos $\mathcal{E}$, the natural twinning

$$\begin{cases} \mathrm{Geom}(\mathcal{E}, \mathcal{E}_{\mathbb{T}}) & \longrightarrow & \mathbb{T}\text{-mod}(\mathcal{E})\,, \\ (\mathcal{E} \xrightarrow{f} \mathcal{E}_{\mathbb{T}}) & \longmapsto & f^* U_{\mathbb{T}}\,, \end{cases}$$

is an equivalence.

Remarks :

(i) The topos $\mathcal{E}_{\mathbb{T}}$ endowed with the model $U_{\mathbb{T}}$ is unique up to equivalence.
The model $U_{\mathbb{T}}$ in $\mathcal{E}_{\mathbb{T}}$ is called the "universal model" of $\mathbb{T}$.

(ii) An implication between two formulas $\varphi(\vec{x}) \vdash \psi(\vec{x})$ is provable in $\mathbb{T}$ if and only if it is verified by $U_{\mathbb{T}}$.

(iii) For any topos $\mathcal{E}$, there are infinitely many theories $\mathbb{T}$ such that

$$\mathcal{E} \cong \mathcal{E}_{\mathbb{T}}\,.$$

Two theories $\mathbb{T}$ and $\mathbb{T}'$ verify the condition

$$\mathcal{E}_{\mathbb{T}} \cong \mathcal{E}_{\mathbb{T}'}$$

if and only if their semantics are equivalent.