Some possible roles for AI

of Grothendieck topos theory

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A basic principle of "deep learning"

- Warning: I know very little on this subject.
  Much of what I know comes from
  a lecture course by Prof. H. Bölskei
  at the Lagrange Center, Paris, in June 2022.
- "Deep learning" is based on a simple idea in functional analysis:
  replace classical "approximation by superposition"
  by "approximation by composition"
- The meaning of "approximation by superposition":
  Approximate functions (in a given functional space)
  by linear combinations
  of elements of a given family of special functions
  (eg: some Hilbert basis such as the family of Fourier characters).
- The meaning of "approximation by composition":
  Approximate functions (on some compact subspace of a f.d. linear space)
  by (finite but arbitrarily long) composites
  of functions which belong to simple special classes.
- Fact found in practice:
  Approximation by composition proves to be more efficient!

A type of composites which appears in practice

- One tries to approximate (continuous) functions

$$f : S \longrightarrow \mathbb{R}$$

on some compact subspace $S \subset \mathbb{R}^n$.

- One looks for approximation by composites of the form

$$f_\ell \circ \varphi_{\ell-1} \circ f_{\ell-1} \circ \cdots \circ f_2 \circ \varphi_1 \circ f_1$$

$$\underset{\mathbb{R}^n=\mathbb{R}^{n_0}}{\bullet} \longrightarrow \underset{\mathbb{R}^{n_1}}{\bullet} \longrightarrow \underset{\mathbb{R}^{n_2}}{\bullet} \longrightarrow \cdots \longrightarrow \underset{\mathbb{R}^{n_{\ell-1}}}{\bullet} \longrightarrow \underset{\mathbb{R}=\mathbb{R}^{n_\ell}}{\bullet}$$

where

- each $f_k : \mathbb{R}^{n_{k-1}} \longrightarrow \mathbb{R}^{n_k}$, $1 \leq k \leq \ell$,
  is an affine function
  whose coefficients have to be optimized,
- each $\varphi_k : \mathbb{R}^{n_k} \longrightarrow \mathbb{R}^{n_k}$, $1 \leq k < \ell$, has the form

$$(x_1, \cdots, x_{n_k}) \longmapsto (\varphi(x_1), \cdots, \varphi(x_{n_k}))$$

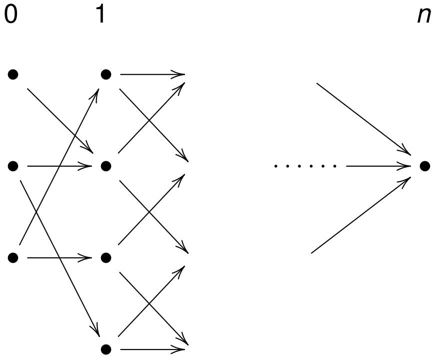
for some given non-linear "truncation" function

$$\varphi : \mathbb{R} \longrightarrow \mathbb{R}$$

(eg: $\varphi(t) = \max\{0, t\}$).

More refined types of composition diagrams

They may take the form of quivers



consisting in

- finitely many dots or objects
  arranged in "layers" 0 ("entries" layer)
  1
  ⋮
  n ("output" layer),
- finitely many arrows
  which go from some layer k
  to layer $k+1$ (or, more generally, $k' > k$).

Induced formalisation (in Belfiore-Bennequin):

- such a quiver generates a finite base category $\mathcal{B}$,
- some natural "product decomposition" conditions
  on linear spaces indexed by the objects of $\mathcal{B}$
  can be expressed as "sheaf conditions" for some topology J on $\mathcal{B}$.

Two foundational problems of machine computing on numerical functions

- First obvious problem:
  Numbers do not have meaning in themselves.
  A fortiori, numerical functions have no meaning in themselves.
- Consequences:
  - When elements of information about the real world (ex: images) are reduced to collections of numbers (ex: pixels) or numerical functions, their meaning disappears.
  - The way they are processed along the layers of deep learning devices is a black box.
  - Replacing meaningful elements of information by collections of numbers or numerical functions puts them in an environment (consisting in big linear spaces of coordinates or numerical functions) where almost all elements do not correspond to anything meaningful, resulting in a huge loss of efficiency.
- Second more subtle problem:
  Numerical computing is perhaps not the best way to use computers.
  Indeed, they are more "logical machines".

Taking into account underlying symmetries

Observation. –

- It often happens that the "entries"
  in a deep learning network
  are naturally submitted to some transformations.

  For instance:

  - images can be translated,
  - an image can fit as a piece of another image.

- It is natural to expect that
  if there are natural symmetries and relations at the entries layer
  there should also exist natural symmetries at deeper layers,
  and the processing arrows between the different layers
  should respect in some way the underlying symmetries.

Proposed formalisation in Belfiore-Bennequin

- The existence of compatible systems of natural symmetries could be formalised by associating

$$\begin{cases} - & \text{to any object } X \text{ of the base category } \mathcal{B} \\ & \text{a category } \mathcal{C}_X, \\ - & \text{to any processing arrow } X \xrightarrow{f} Y \text{ of } \mathcal{B} \\ & \text{a functor } f^* : \mathcal{C}_Y \longrightarrow \mathcal{C}_X. \end{cases}$$

- The categories $\mathcal{C}_X$ related by these functors can be understood as the fibers over the objects of $\mathcal{B}$ of a "fibered category"

$$\mathcal{C} \xrightarrow{p} \mathcal{B},$$

i.e. a category $\mathcal{C}$ endowed with a projection functor

$$p : \mathcal{C} \longrightarrow \mathcal{B}$$

which verifies some special properties.

When is a number or a numerical function meaningful?

- It was first considered that
  linear spaces of coordinates
  and transformations between them
  had to be associated to
  the objects and arrows of the base category $\mathcal{B}$.
- Trying to take into accounts natural symmetries
  leads to the idea of associating
  linear spaces of coordinates and transformations between them
  to
  the objects and arrows of the fibered category $\mathcal{C}$ over $\mathcal{B}$.
- But we are left with the question:
  When does a vector in a linear space of coordinates
  have meaning?

Coming back to the definition of numbers:
A number has meaning when it appears as
the number of elements of a concrete finite set!

Some lesson of arithmetic algebraic geometry

- It can be argued that arithmetic algebraic geometers only consider functions which, for them, have meaning.
- The meaning of the functions they consider consists in the fact that their values or coefficients are numerical invariants of geometric objects.

Examples. –

- The numbers of points over finite extensions $\mathbb{F}_{q^n}$ of algebraic varieties defined over finite fields $\mathbb{F}_q$.
- The numbers of fixed points of geometric morphisms or correspondences (and their composites) from some algebraic varieties (or schemes) to themselves.
- The "traces" of geometric morphisms and correspondences acting on cohomology spaces of algebraic varieties or schemes.

Traces

- The word "trace" has a precise math meaning in linear algebra:
  - the sum of the diagonal entries of a square matrix,
  - the sum of the eigenvalues of an endomorphism of a finite dimensional vector space.
- One can also remember about the meaning of "trace" in natural language:
  - For instance, the "traces" of an animal that walked on the ground.
  - Such traces make sense for us when we understand that they come from an animal, that is the real being.
  - Most often, we don't see wild animals, we only see their traces and get partial information about them through their traces.
- The same in mathematics:
  - Numbers, numerical functions or other types of invariants make sense when they are the "traces" of (i.e. are defined from) "real mathematical objects", that are geometric spaces.
  - Most often, we don't fully know and understand geometric objects, but we get partial information about them through their "traces" which are numerical or non-numerical "invariants".

The problem of lifting function-theoretic operations to geometry

- Numbers and numerical functions are of much use because they can be transformed by operations such as

  - addition,
  - multiplication,
  - change of parameters, composition,
  - taking limits,
  - integration, convolution,
  - application of functionals on function spaces.

- Only considering numbers or numerical functions of geometric origin, and keeping track of the geometric objects they come from, allows to preserve meaning, but makes more difficult or impossible to make use of all function-theoretic operations: some of them cannot be lifted at the level of classical notions of spaces.

The solution provided by sheaf-theoretic completions

- Usually, any math environment of geometric nature
  can be seen as a category

  $$\mathcal{G}$$

  endowed with a notion of local-global duality
  mathematically defined as a Grothendieck topology J.

- The topos $\widehat{\mathcal{G}}_J$ of J-sheaves on $\mathcal{G}$ is endowed with a canonical functor

  $$\ell : \mathcal{G} \longrightarrow \widehat{\mathcal{G}}_J \text{ (defined as } \mathcal{G} \xrightarrow{\text{Yoneda}} \widehat{\mathcal{G}} \xrightarrow{\text{sheafification}} \widehat{\mathcal{G}}_J)$$

  so that
  $\widehat{\mathcal{G}}_J$ can be seen as a "completion" of $\mathcal{G}$
  where all usual set-theoretic constructions exist universally
  and make possible
  to lift all usual function-theoretic operations
  and even to supplement them with
  more refined "derived" operations
  which could not be defined in function-theoretic terms.

Operations on objects of toposes

- Inner operations inside a topos:
  - Non-linear sums $\coprod$ and linear direct sums $\oplus$.
  - Non-linear products $\times$ and linear tensor products $\otimes$.
  - Non-linear and linear exponentiations $\mathcal{H}om(\bullet, \bullet)$.
- Outer operations between two toposes related by a geometric morphism $f : \mathcal{E}' \to \mathcal{E}$:
  - Pull-back functor $f^* : \mathcal{E} \to \mathcal{E}'$ (which lifts "change of parameters").
  - Its right adjoint $f_* : \mathcal{E}' \to \mathcal{E}$ (= functor of relative globalisation), plus the "cohomology" functors of f_* acting on linear sheaves.
  - A "homology" functor $f_!$ acting on complexes of linear sheaves.
  - Its right-adjoint $f^!$ endowed with "integration" $f_! \circ f^! \to \mathrm{Id}$.
- Expected notion of "constructible sheaf" in $\widehat{\mathcal{G}}_J$ such that:
  - All sheaves coming from objects of $\mathcal{G}$ through $\mathcal{G} \xrightarrow{\ell} \widehat{\mathcal{G}}_J$ are constructible.
  - Constructible sheaves are stable under all above operations.
  - The natural numerical or function-theoretic invariants of geometric objects in $\mathcal{G}$ remain well-defined for all constructible sheaves.

The problem of classifying "meaningful data"

- So far, math experience (especially in algebraic geometry) gave us the idea that numerical data have meaning when they arise from geometric objects (such as "constructible sheaves") as "invariants" associated with them.
- In order to preserve meaning throughout the data processing, one would dream of replacing
  the linear spaces of data coordinates
  (where most elements have no meaning)

  by

  spaces of the type of geometric objects
  which give meaning in the situation under consideration.
- For instance, is it possible to mathematically define
  a "space of images", such that
  - any image can be seen as a point of this space,
  - any point of this space defines an image which could actually appear.

The alternate problem of theorizing "meaningful data"

- Alternatively, one can wonder whether it would be possible
  to give a linguistic description
  of
  the type of geometric objects
  which give meaning in the situation under consideration.
- Such a linguistic description should consist in
  - a rich enough vocabulary, i.e. a list of names
    for all constituents and structures
    of the type of geometric objects which may appear,
  - a list of properties
    (formulated in terms of the description vocabulary)
    that would characterize
    the type of geometric objects which may appear.
- In other words, one would look for
  a theory of the type of geometric objects under consideration.
- For instance,
  is it possible to formulate
  a theory (or theories) of images?

Which type of classifying spaces?

Starting pinciple:
One should not decide a priori the type of geometric structure that may exist on classifying spaces.
Anyway, it should depend on the type of “meaningful objects” one tries to classify.

General observation:
All types of “geometric spaces” which have been introduced in classical or contemporary mathematics can be seen as toposes generally endowed with an extra structure:

- The underlying topos is the topological component.
- The more refined geometric structure is usually incarnated in an “inner structure” of the topos, most often an inner ring ($=$ sheaf of rings whose sections can be seen as coordinate functions).

Natural conclusion:
Looking for a “classifying space” of meaningful objects under consideration, one can look first for a topos, then wonder whether it is endowed with natural structures.

Which type of language description for "meaningful objects"?

Obvious requirement:

As we are aiming for computer implementation, we need to restrict to

- language descriptions in the mathematical sense, i.e. theories in the sense of logic,
- constructive mathematics.

Observation:

Math. constructions in the context of sets can all be phrased in terms of

- conjunctions $\wedge$, $\top$ (truth), $\bigwedge$,
- disjunctions $\vee$, $\perp$ (falsity), $\bigvee$,
- existential and universal quantifiers $\exists$, $\forall$,
- negation $\neg$,
- implication $\Rightarrow$,
- exponentiations $B^A = \mathcal{H}om(A, B) = \{\text{arrows } A \to B\}$, and $\Omega^A = \{\text{subobjects } S \hookrightarrow A\}$.

General form of a constructive math theory: It consists in

- a list of names for constituents (objects, arrows, relations) of its "models",
- axioms (phrased with these names and the above symbols) which its "models" have to verify.

Which type of theories of "meaningful objects"?

General observations:

- Any constructive math. theory $\mathbb{T}$ has set-theoretic models.
- As any topos $\mathcal{E}$ has all constructive categorical properties of Set, such a theory $\mathbb{T}$ defines as well

$$\mathbb{T}\text{-models in } \mathcal{E}.$$

- The $\mathbb{T}$-models in a topos $\mathcal{E}$ can be considered as "continuous families" of $\mathbb{T}$-models parametrized by $\mathcal{E}$ if

– any point $\mathrm{Set} \xrightarrow{p} \mathcal{E}$ defines an instantiation operation

$$p^* : \mathbb{T}\text{-models in } \mathcal{E} \longrightarrow \mathbb{T}\text{-models in } \mathrm{Set},$$

– more generally, toposes morphisms $\mathcal{E}' \xrightarrow{f} \mathcal{E}$ define operations of change of parameters

$$f^* : \mathbb{T}\text{-models in } \mathcal{E} \longrightarrow \mathbb{T}\text{-models in } \mathcal{E}'.$$

Key fact about toposes. – *The last requirement is fulfilled if (and only if) $\mathbb{T}$ is a "first-order geometric theory" in the sense that*

- *its formulation doesn't make use of exponentiations B^A, Ω^A,*
- *its axioms can be phrased with the logical symbols*

$$\wedge, \top \text{ (truth)}, \bigvee, \bot \text{ (falsity)}, \exists.$$

Classifying toposes and linguistic description theories

- Any "first-order geometric" theory $\mathbb{T}$ associates
  - to any topos $\mathcal{E}$, a category of $\mathbb{T}$-models in $\mathcal{E}$ $\quad \mathbb{T}\text{-mod}\,(\mathcal{E})$,
  - to any toposes morphism $\mathcal{E}' \xrightarrow{f} \mathcal{E}$, a functor of "change of parameters"
    $$f^* : \mathbb{T}\text{-mod}\,(\mathcal{E}) \longrightarrow \mathbb{T}\text{-mod}\,(\mathcal{E}').$$

Theorem. –

(i) *Any presentation of a topos $\mathcal{E}$ as the topos of sheaves on some presenting site $(\mathcal{G}, J)$*

$$\mathcal{E} \xrightarrow{\sim} \widehat{\mathcal{G}}_J$$

defines a first-order geometric theory $\mathbb{T}_{\mathcal{G},J}$ which provides a linguistic description of generalized points of $\mathcal{E}$

$$\{\text{category of toposes morphism } \mathcal{E}' \to \mathcal{E}\} \xrightarrow{\sim} \mathbb{T}_{\mathcal{G},J}\text{-mod}\,(\mathcal{E}').$$

(ii) *Conversely, any first-order geometric theory $\mathbb{T}$ has a "classifying topos" $\mathcal{E}_{\mathbb{T}}$ endowed with a "universal $\mathbb{T}$-model" $M_{\mathbb{T}}$ which induces equivalences*

$$\{\text{category of toposes morphism } \mathcal{E} \to \mathcal{E}_{\mathbb{T}}\} \xrightarrow{\sim} \mathbb{T}\text{-mod}\,(\mathcal{E}),$$
$$(\mathcal{E} \xrightarrow{f} \mathcal{E}_{\mathbb{T}}) \longmapsto f^* M_{\mathbb{T}} \quad .$$

The correspondence of topology and language through toposes

- According to the previous theorem,
  it is theoretically equivalent to
  - look for a Grothendieck topos
    which would be the underlying topological structure
    of a “classifying space”
    for a type of “meaningful objects”
    lifting a type of “meaningful data” under consideration,
  - look for “first-order geometric theories”
    that would describe at least part of the structure
    of “meaningful objects” of geometric nature
    that would incarnate the “meaning”
    of data under consideration.

A duality of objects and points of some toposes?

Observation. –

- *At first, inspired by the experience of algebraic geometry, we introduced the idea that numerical data could have meaning when they appear as "invariants" of some particular classes of objects ("constructible sheaves") in some toposes.*
- *Secondly, hoping to classify these particular classes of objects which would incarnate meaning, we introduced the idea that they could also appear as points of some other toposes.*

Commentary: This is not absurd.
For instance, sets endowed with an action of a group G can be interpreted simultaneously as

- the objects of the "classifying topos" of the group G,
- the points of the topos which classifies the theory of actions of G.

Networks of toposes?

- As in Belfiore-Bennequin, consider a finite base category $\mathcal{B}$ lifted to a fibered category of structural symmetries $p : \mathcal{C} \longrightarrow \mathcal{B}$.
- One can imagine that any object X' of $\mathcal{C}$ could be associated with a notion of "meaningful geometric objects" at X' that would incarnate the meaning of the type of data which would appear at step X' of the processing network.
  - If these "objects incarnating meaning" are geometric, they would appear as particular classes of objects in a topos $\mathcal{E}^1_{X'}$.
  - They could be "classified" if they could be interpreted as points of another "dual" topos $\mathcal{E}^2_{X'}$.
- One can also imagine that any morphism $X' \xrightarrow{f'} Y'$ of $\mathcal{C}$ could be associated with a transformation operation

$$\{\text{"meaningful geometric objects" at } X'\} \to \{\text{"meaningful geometric objects" at } Y'\}.$$

  - This transformation process would be meaningful itself if it could be defined as a Grothendieckian operation on sheaves

$$\{\text{objects in } \mathcal{E}^1_{X'}\} \longrightarrow \{\text{objects in } \mathcal{E}^2_{X'}\}.$$

  - If it also acted on "continuous families" of objects, it would lift to a morphism of toposes $\mathcal{E}^2_{X'} \longrightarrow \mathcal{E}^2_{Y'}$.

In summary: why topos completions?

- Math experience, especially in algebraic geometry, gives the idea that the meaning of (numerical) data could be incarnated in geometric objects from which these data would be deduced as (numerical) "invariants".
- Geometric objects of some given type X' always make up a category $\mathcal{G}_{X'}$, generally endowed with a Grothendieck topology $J_{X'}$.
- Replacing such a geometric category $\mathcal{G}_{X'}$ by its topos completion

$$\mathcal{G}_{X'} \hookrightarrow \widehat{(\mathcal{G}_{X'})}_{J_{X'}} = \mathcal{E}^1_{X'}$$

allows to define in full generality natural sheaf-theoretic operations

$$\{\text{objects of } \mathcal{E}^1_{X'}\} \longrightarrow \{\text{objects of } \mathcal{E}^1_{Y'}\}$$

that can lift any natural function-theoretic operation one would want to associate to a transformation arrow

$$X' \longrightarrow Y'.$$

In summary: why classifying toposes?

- At each step X' of the processing network, there should be a notion of "meaningful geometric object" that would incarnate the "meaning" of the type of data which would appear at X'.
- Such a notion should belong to constructive mathematics. In other words, there should be

  mathematical theories $\mathbb{T}_{X'}$

  of such a type of geometric objects.
- These theories or simplified versions of them should allow to define

  "continuous families" of such geometric objects

  and

  functors of "change of parameters"

  for such "continuous families".
- This would be enough to define

  "classifying toposes" $\mathcal{E}^2_{X'}$

  verifying the property that

  the types of geometric objects under consideration

  appear as

  points of these toposes $\mathcal{E}^2_X$.