Moduli problems, classifying toposes, invariants, and the six operations formalism

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Plan:

The general problem of classification

Given a certain type of "math. structures" or "natural data",
one may look for answers to the related three questions:

1) Elaborate an appropriate language
(consisting in a vocabulary and grammar rules)
for describing the type of data under consideration, such as
any type of mathematical structures,
images, chess games, texts, $\cdots$

2) Find out whether these data make up a
space
endowed with a natural
topological structure
possibly enriched with a
geometric structure.

3) Find efficient ways to represent
points of such classifying spaces.

Examples of classification problems in mathematics

- Classification of lines (containing the origin 0) in a n-dimensional vector space.
- Classification of r-dim. subspaces in a n-dimensional vector space.
- Classification of closed varieties (of algebraic or analytic nature) of a given variety.
- Classification of algebraic curves (considered abstractly, without embedding).

Remark:
One begins with classification problems in mathematics,
because mathematics is easier:
mathematical structures are easier to define.

Comparison of situations of mathematicians and AI theoricians

For mathematicians:

- The notion of line is clear and easy to define.
- The notion of r-dim. subspace is well-defined
  as soon as the notion of n-dim. space is defined.
- The notion of (algebraic or analytic) subvariety of a given variety
  is natural
  but it took some time to find fully satisfactory definitions.
- The notion of abstract curve (or of abstract variety)
  took even more time to be fully well-defined.

For AI theoricians:

- A general definition of a chess game is easy to give,
  even though almost all games verifying such a definition
  would be much worse then the worse games played by humans.
- We all know what is an image, but we have no general formal definition.
- We all know what is a meaningful text, but we have no formal definition.

First example: classification of lines

1) Any line in a vector space V can be presented as generated by a non-zero vector $v \in V$.

2) The space of possible presentations of lines is identified as the open subspace
$$P = V - \{0\} \hookrightarrow V\,.$$

3) Two non-zero vectors generate the same line if and only if they belong to the same orbit under the group action
$$\begin{array}{rcl} \mathrm{GL}_1 \times (V - \{0\}) & \longrightarrow & V - \{0\}\,, \\ (\lambda, v) & \longmapsto & \lambda \cdot v\,. \end{array}$$

4) The classifying space of lines is constructed as the quotient
$$\varinjlim \,[\mathrm{GL}_1 \times (V - \{0\}) \rightrightarrows V - \{0\}]$$
$$= (V - \{0\})/\mathrm{GL}_1$$
$$= \mathbb{P}_V \qquad (\text{"projective space" of } V).$$

Second example: classification of linear subspaces

1) Any r-dim. subspace of a given n-dim. space V

$$S \hookrightarrow V$$

can be presented by a family of r linearly independent vectors of V.

2) The space of possible presentations of r-dim. subspaces of V is identified as an open subspace

$$P \hookrightarrow (V - \{0\})^r \quad .$$

3) Two families of r linearly independent vectors generate the same subspace if and only if they belong to the same orbit under the group action

$$\mathrm{GL}_r \times P \longrightarrow P\,,$$

$$\big(M = (m_{i,j})_{1 \le i,j \le r}; (v_1, \cdots, v_r)\big) \longmapsto \left(\sum_{j=1}^{r} m_{i,j} \cdot v_j\right)_{1 \le i \le r} \quad .$$

4) The classifying space of r-dim. subspaces is constructed as the quotient

$$P/\mathrm{GL}_r = \mathrm{Grass}_V^r \quad \text{("Grassmannian" of degree } r \text{ of } V).$$

Complement:

The linear algebra construction $\quad (S \hookrightarrow V) \mapsto (\text{line } \Lambda^r S \hookrightarrow \Lambda^r V)$
defines an embedding as a subvariety $\quad \mathrm{Grass}_V^r \hookrightarrow \mathbb{P}_{\Lambda^r V}$
determined by polynomial equations called the "Plücker coordinates".

Third example: classification of projective algebraic subvarieties

Start with the projective space $\mathbb{P}^n$ of the affine space $\mathbb{A}^{n+1}$.

Let V_d = space of homogeneous polynomials of degree d on $\mathbb{A}^{n+1}$.

1) For any subvariety
$$X \hookrightarrow \mathbb{P}^n \,,$$
show that if d is big enough, the subspace
$$I_X = \{P \in V_d \mid P \text{ vanishes on } X\}$$
has dimension $r = \chi_X(d)$ for χ_X = "Hilbert-Poincaré polynomial" of X and I_X determines X.

2) Deduce that subvarieties $X \hookrightarrow \mathbb{P}^n$ such that $\chi_X = \chi$ (= fixed polynomial) can be presented by elements of a subspace
$$\mathrm{Hilb}^{\chi}_{\mathbb{P}^n} \hookrightarrow \mathrm{Grass}^{r}_{V_d} \qquad (\text{"Hilbert variety"})$$
defined by polynomial equations.

3) Show that the abstract algebraic variety
$$\mathrm{Hilb}^{\chi}_{\mathbb{P}^n}$$
does not depend on the choice of the degree d (if d is big enough).

4) Define the classifying algebraic variety of algebraic varieties
$$X \hookrightarrow \mathbb{P}^n \quad \text{such that} \quad \chi_X = \chi$$
as
$$\mathrm{Hilb}^{\chi}_{\mathbb{P}^n}.$$

Fourth example: classification of algebraic curves

1) Show that any abstract algebraic curve C of genus g with a marked point c admits presentations by embeddings of C of degrees $d \gg g$
$$C \hookrightarrow \mathbb{P}^n \quad \text{with} \quad n = d - g\,.$$

2) Show that possible presentations of curves of genus g make up a subspace
$$P \hookrightarrow \mathrm{Hilb}^{\chi}_{\mathbb{P}^n} \qquad (\text{for } \chi = X + 1 - g).$$

3) Show that the automorphism group $\mathrm{PGL}_n = \mathrm{Aut}(\mathbb{P}^n)$ acts on
$$P \hookrightarrow \mathrm{Hilb}^{\chi}_{\mathbb{P}^n}$$
and that two elements of P represent the same abstract algebraic curve C if and only if they belong to the same orbit.

4) Define the classifying space of abstract algebraic curves as the quotient
$$P/\mathrm{PGL}_n = \mathcal{M}_{g,1} \quad (\text{“Moduli space of curves of genus } g \text{ with a marked point”}).$$

Summary: classical constructions of classifying spaces

After defining the type of structures to be classified,
mathematicians usually made the following steps:

1) Show that any structure of the chosen type can be presented by some concrete data (which, in general, are not unique).

2) Show that the type of data which may appear as presentations make up a subspace P of some already familiar ambient space E, identify the relations which define this subspace

$$P \hookrightarrow E$$

and endow it with a geometric (algebraic or analytic) structure induced from the ambient space E.

3) Compute the relation subspace $R \hookrightarrow P \times P$ of pairs of presentations which define the same object. (Most often, $R = G \times P$ for a group G acting on P.)

4) Define the "classifying space" as a "quotient", i.e. a "colimit" of the diagram $R \rightrightarrows P$, endowed with the geometric (algebraic or analytic) structure induced from the presentation space P.

Grothendieck's definition of classifying spaces

Key new starting point:
Consider not only the objects to be classified but natural "continuous" families of such objects parametrized by a natural type of spaces.

Mathematical set-up:

- Possible natural spaces S for parametrizing the objects under consideration, together with the maps respecting their natural structures
$$S' \longrightarrow S$$
should make up a category $\mathcal{C}$.
- A definition of a general notion of continuous families of objects of type $\mathbb{T}$ should allow to associate to any parameter space S in $\mathcal{C}$
a set $\quad M(S)$
of all continuous families of objects of type $\mathbb{T}$ parametrized by S.
- This definition should be good enough to associate to any map of parametrizing spaces
$$S' \longrightarrow S \qquad (= \text{arrow of the category } \mathcal{C})$$
an operation of "change of parameters" which is
a map $\quad M(S) \longrightarrow M(S')$.
- Change of parameters should be compatible with compositions
$$\begin{tikzcd} (S'' \arrow[r] \arrow[rr, bend right] & S' \arrow[r] & S) \end{tikzcd} \Rightarrow \begin{tikzcd} (M(S) \arrow[r] \arrow[rr, bend right] & M(S') \arrow[r] & M(S'')). \end{tikzcd}$$

Examples from geometry

Grothendieck defined

- a notion of algebraic family of lines in a vector space V parametrized by an algebraic variety S,
- a notion of algebraic family of r-dim. subspaces of a vector space V parametrized by an algebraic variety S,
- a notion of algebraic family of subvarieties
$$X \hookrightarrow \mathbb{P}^n$$
parametrized by an algebraic variety S
(as a closed subvariety
$$X \hookrightarrow \mathbb{P}^n \times S$$
whose projection morphism $X \to S$ is "flat"),
- $\cdots$

allowing to define sets
$$M(S) = \{\text{algebraic families parametrized by } S\}$$
related by maps of "change of parameters"
$$M(S) \longrightarrow M(S')$$
defined for all arrows
$$S' \longrightarrow S$$
of the category $\mathcal{C}$ of algebraic varieties.

Formal definition of classification problems in Grothendieck's sense

Definition (Grothendieck). –
A classification problem consists in

- *a category $\mathcal{C}$ (supposed to be small or essentially small) whose objects S are seen as "parametrizing spaces" and whose arrows $S' \to S$ are seen as "changes of parameters",*
- *a "presheaf", i.e. a contravariant functor*

$$\begin{cases} M \quad : & \mathcal{C}^{\mathrm{op}} & \longrightarrow & \mathrm{Set}, \\ & \textit{object } S & \longmapsto & M(S) = \mathrm{set}, \\ & \textit{arrow } (S' \to S) & \longmapsto & (M(S) \to M(S')) = \textit{map}. \end{cases}$$

Reminder:
A category can be considered as some kind of "mathematical country" consisting in

- "objects" = cities of some type (usually the models of a theory),
- "arrows" = itineraries between pairs of cities (usually maps which respect model structures),
- a composition law of arrows or itineraries

$$(X \xrightarrow{f} Y \xrightarrow{g} Z) \longmapsto (X \xrightarrow{g \circ f} Z)$$

which is associative

and has a "unit arrow" $(X \xrightarrow{\mathrm{id}_X} X)$ associated to any object X.

Grothendieck's definition of classifying spaces

Definition (Grothendieck). –

Consider a classification problem consisting in

- *a category $\mathcal{C}$ of "parametrizing spaces",*
- *a presheaf (or contravariant functor)*
  $$M_{\mathbb{T}} : \mathcal{C}^{\mathrm{op}} \longrightarrow \mathrm{Set}$$
  which associates wtih any "parametrizing space" S the set
  $$M_{\mathbb{T}}(S)$$
  of "continuous families" of structures of some type $\mathbb{T}$ parametrized by S.

Then an object $S_{\mathbb{T}}$ of $\mathcal{C}$ is called a "classifying space" for structures of type $\mathbb{T}$ if there exists an element

$$m_{\mathbb{T}} \in M_{\mathbb{T}}(S_{\mathbb{T}})$$

which is "universal" in the sense that, for any object S of $\mathcal{C}$, the map

$$\begin{array}{rcl} \text{(arrow } S \xrightarrow{s} S_{\mathbb{T}} \text{ in } \mathcal{C}) & \longmapsto & s^*(m_{\mathbb{T}}) \in M_{\mathbb{T}}(S)\,, \\ \mathrm{Hom}(S, S_{\mathbb{T}}) & \longrightarrow & M_{\mathbb{T}}(S) \end{array}$$

is one-to-one.

Determination of classifying spaces with their structures

Proposition. –

Consider a classification problem in Grothendieck's sense

$$(\mathcal{C}, M_{\mathbb{T}} : \mathcal{C}^{\mathrm{op}} \longrightarrow \mathrm{Set}) \,.$$

Then, if there exists an associated "classifying space"

$$S_{\mathbb{T}} = \textit{object of } \mathcal{C},$$

it is unique up to unique isomorphism in $\mathcal{C}$.

Remark:

In particular, its structure as an object of $\mathcal{C}$

(which, usually, consists in geometric objects of some type)

is determined by the associated presheaf

$$S \longmapsto M_{\mathbb{T}}(S) = \mathrm{Hom}(S, S_{\mathbb{T}}) \,,$$

i.e. by the general notion of

"continuous families of structures of type $\mathbb{T}$"

parametrized by objects of $\mathcal{C}$.

The reason why classifying spaces are uniquely determined

This is thanks to the following result
which is very easy but the most important result of category theory.

Yoneda's lemma. –
Consider a category $\mathcal{C}$ which is small or essentially small.
Associate to any object X of $\mathcal{C}$ the presheaf

$$y(X) = \mathrm{Hom}(\bullet, X) \quad : \quad \begin{array}{rcl} \mathcal{C}^{\mathrm{op}} & \longrightarrow & \mathrm{Set}, \\ S & \longmapsto & \mathrm{Hom}(S, X)\,. \end{array}$$

Consider the category $\widehat{\mathcal{C}}$ of presheaves

$$P : \mathcal{C}^{\mathrm{op}} \longrightarrow \mathrm{Set}.$$

Then:

(i) *For any object X of $\mathcal{C}$ and presheaf P in $\widehat{\mathcal{C}}$*

$$\mathrm{Hom}_{\widehat{\mathcal{C}}}(y(X), P) \quad \underline{\textit{identifies with}} \quad P(X)\,.$$

(ii) *In particular, for any objects S, X of $\mathcal{C}$*

$$\mathrm{Hom}_{\widehat{\mathcal{C}}}(y(S), y(X)) \quad \underline{\textit{identifies with}} \quad \mathrm{Hom}(S, X)\,.$$

Remark:

The Yoneda functor

$$y : \mathcal{C} \hookrightarrow \widehat{\mathcal{C}} \qquad \text{is fully faithful.}$$

Conditions for the existence of classifying spaces

Problem: *Given a classification problem*

$$(\mathcal{C}, M_{\mathbb{T}} : \mathcal{C}^{\text{op}} \longrightarrow \text{Set}),$$

is it possible to formulate conditions for the existence of an associated classifying space?
In other words, are there conditions for a presheaf

$$P : \mathcal{C}^{\text{op}} \longrightarrow \text{Set}$$

to be isomorphic to the image of an object X of $\mathcal{C}$ by Yoneda

$$y : \mathcal{C} \hookrightarrow \widehat{\mathcal{C}} \quad ?$$

Formulation of a necessary condition:

(i) *On any small or essentially small category $\mathcal{C}$, there is*

- *a notion of "Grothendieck topology" on $\mathcal{C}$,*
- *a property of presheaf $P : \mathcal{C}^{\text{op}} \longrightarrow \text{Set}$ to be or not to be a "sheaf" with respect to any given topology J.*

(ii) *A topology J on $\mathcal{C}$ may be "subcanonical" in the sense that, for any object X of $\mathcal{C}$, $y(X) = \text{Hom}(\bullet, X)$ is a J-sheaf.*

(iii) *If J is a subcanonical topology, a classification problem*

$$(\mathcal{C}, M_{\mathbb{T}} : \mathcal{C}^{\text{op}} \longrightarrow \text{Set})$$

can define a "classifying space" only if $M_{\mathbb{T}}$ is a J-sheaf.

Preliminaries on sieves and presieves

Definition. – *Let $\mathcal{C}$ be a category which is small or essentially small.*

(i) *A presieve on an object X is a family of arrows*

$$X_i \xrightarrow{x_i} X\,, \quad i \in I\,.$$

(ii) *A sieve on an object X is a subobject in $\widehat{\mathcal{C}}$*

$$C \hookrightarrow \mathrm{Hom}(\bullet, X) = y(X)$$

or, equivalently, a collection of arrows

$$X' \xrightarrow{x} X$$

which is stable under composition on the right with arbitrary arrows

$$(X'' \xrightarrow{x'} X') \longmapsto (X'' \xrightarrow{x \circ x'} X).$$

Remarks:

(1) Any presieve $(X_i \xrightarrow{x_i} X)_{i \in I}$ generates the sieve

$$\{X' \xrightarrow{x} X \mid \exists\, i \in I\,,\ \exists (X' \xrightarrow{x'} X_i)\,,\ x = x_i \circ x'\}.$$

(2) Any arrow in $\mathcal{C}$

$$S \xrightarrow{x} X$$

transforms sieves on X

$$C \hookrightarrow \mathrm{Hom}(\bullet, X)$$

into sieves on S

$$x^* C = \{S' \xrightarrow{s} S \mid x \circ s \in C\}.$$

The sheaf condition

Definition. –
Let $\mathcal{C}$ be a small or essentially small category.
One says that a presheaf

$$P : \mathcal{C}^{\mathrm{op}} \longrightarrow \mathrm{Set}$$

respects the "sheaf condition" with respect to a sieve on an object X

$$C \hookrightarrow \mathrm{Hom}(\bullet, X)$$

if, for any arrow

$$X' \xrightarrow{x} X,$$

the restriction map

$$P(X') = \mathrm{Hom}(y(X'), P) \longrightarrow \mathrm{Hom}(x^*C, P)$$

is one-to-one.

Remarks:

(Max) Maximal sieves

$$C = \mathrm{Hom}(\bullet, X)$$

verify the condition for any presheaf P.

(Stab) By definition, if a sieve

$$C \hookrightarrow \mathrm{Hom}(\bullet, X)$$

verifies the condition with respect to some presheaves P, so do the sieves

$$x^*C \hookrightarrow \mathrm{Hom}(\bullet, X') \text{ for any } X' \xrightarrow{x} X.$$

Grothendieck topologies

Theorem. –
Start with an arbitrary collection $\mathcal{P}$ of presheaves on $\mathcal{C}$.
Let J be the collection of sieves on objects X of $\mathcal{C}$

$$C \hookrightarrow \mathrm{Hom}(\bullet, X)$$

which verify the sheaf condition with respect to elements P of $\mathcal{P}$.
Then J is a Grothendieck topology in the sense that,
in addition to

$$\text{(Max)} \quad \textit{and} \quad \text{(Stab)},$$

it respects the "locality" axiom:

(Loc) *In order for a sieve*

$$C \hookrightarrow \mathrm{Hom}(\bullet, X)$$

to belong to J, it suffices that there exists a presieve on X

$$(X_i \xrightarrow{x_i} X)_{i \in I}$$

such that

$$\begin{cases} \bullet \quad \text{the sieve on } X \text{ generated by } (X_i \xrightarrow{x_i} X)_{i\in I} \text{ belongs to } J, \\ \bullet \quad \text{for any } i \in I,\ x_i^* C \text{ belongs to } J. \end{cases}$$

Full categories of sheaves

Definition. –
For any topology J on $\mathcal{C}$, the full subcategory

$$\widehat{\mathcal{C}}_J \hookrightarrow \widehat{\mathcal{C}}$$

of presheaves which respect the sheaf condition with respect to elements of J is called the topos of J-sheaves on $\mathcal{C}$.

Remark:
The collection of sieves $C \hookrightarrow \mathrm{Hom}(\bullet, X)$
which respect the sheaf condition with respect to all J-sheaves
is none other than J.

Theorem. –
Let J be a topology on $\mathcal{C}$.

(i) *The embedding functor*

$$j_* : \widehat{\mathcal{C}}_J \hookrightarrow \widehat{\mathcal{C}}$$

has a left adjoint, called the "sheafification functor"

$$j^* : \widehat{\mathcal{C}} \longrightarrow \widehat{\mathcal{C}}_J.$$

(ii) *The natural transform $j^* \circ j_* \to \mathrm{id}_{\widehat{\mathcal{C}}_J}$ is an isomorphism.*

(iii) *The functor j^* respects finite limits in addition to colimits.*

Examples of subcanonical topologies

Theorem (Grothendieck). –

Let $\mathcal{C}$ be the category of algebraic varieties over a base field or, more generally, finitely presentable schemes over a base scheme. Then:

(i) *The "flat topology" on $\mathcal{C}$ is subcanonical.*

(ii) *A fortiori, the Zariski topology or the étale topology are subcanonical.*

Corollary:

A classification problem on such a category $\mathcal{C}$

$$M_{\mathbb{T}} : \mathcal{C}^{\mathrm{op}} \longrightarrow \mathrm{Set}$$

can define a "classifying space" only if

$$M_{\mathbb{T}} \textit{ is a sheaf for the flat topology.}$$

Remark:

If such a $M_{\mathbb{T}}$ is not a sheaf, it can be replaced by its sheafification

$$j^* M_{\mathbb{T}}.$$

Sometimes, it is enough to get a "classifying space".

Definition. –

(i) *If J is a topology on a small or essentially small category $\mathcal{C}$, the topos of J-sheaves on $\mathcal{C}$ is the full subcategory*

$$\widehat{\mathcal{C}}_J \hookrightarrow \widehat{\mathcal{C}}$$

consisting in the presheaves $P : \mathcal{C}^{\mathrm{op}} \to \mathrm{Set}$ *which verify the "sheaf condition" with respect to all sieves*

$$C \hookrightarrow \mathrm{Hom}(\bullet, X) \quad \textit{which belong to } J.$$

(ii) *A Grothendieck topos is a category which is equivalent to* $\widehat{\mathcal{C}}_J$ *for some "sites"* $(\mathcal{C}, J)$.

Reminder: A topology J on $\mathcal{C}$ is a property (called "J-covering") of sieves $C \hookrightarrow \mathrm{Hom}(\bullet, X)$
or, equivalently, of presieves $(X_i \xrightarrow{x_i} X)_{i\in I}$
which only depends on the generated sieves, such that:

$$\begin{cases}
\text{(Max)} & \text{For any } X,\ X \xrightarrow{\mathrm{id}_X} X \text{ is } J\text{-covering.} \\
\text{(Stab)} & \text{If a sieve } C \hookrightarrow \mathrm{Hom}(\bullet, X) \text{ is } J\text{-covering,} \\
 & x^*C \hookrightarrow \mathrm{Hom}(\bullet, X') \text{ is } J\text{-covering for any } X' \xrightarrow{x} X. \\
\text{(Loc)} & \text{If } (X_i \xrightarrow{x_i} X)_{i\in I} \text{ is } J\text{-covering} \\
 & \text{and } (X_{i,j} \xrightarrow{x_{i,j}} X_i)_{j\in I_i} \text{ is } J\text{-covering for any } i \in I, \\
 & \text{then } (X_{i,j} \xrightarrow{x_i \circ x_{i,j}} X)_{\substack{i\in I \\ j\in I_i}} \text{ is } J\text{-covering.}
\end{cases}$$

Toposes as generalized spaces

Interpretation:
Grothendieck considered toposes as generalized spaces.
He even considered that toposes are the true objects of study of "topology".
He chose the word "topos" in particular to mean that "topology" = science of toposes.

Justification:

(1) Any topological space X defines a topos

$$\mathcal{E}_X = \text{topos of sheaves on } \begin{cases} \mathcal{C}_X & = & \text{category of open subsets of } X, \\ J_X & = & \text{usual notion of covering } U = \bigcup\limits_{i \in I} U_i. \end{cases}$$

If X is "sober" (a property almost always verified in practice), it can be recovered from $\mathcal{E}_X$.

(2) There is a notion of "morphism of toposes" $\quad \mathcal{E}' \to \mathcal{E}$

such that any continuous map $\quad X \xrightarrow{f} Y$
defines a topos morphism $\quad \mathcal{E}_X \to \mathcal{E}_Y$.

If Y is sober, $(X \xrightarrow{f} Y) \longmapsto (\mathcal{E}_X \to \mathcal{E}_Y)$ is one-to-one.

(3) There is a notion of "subtopos" of a topos $\quad \mathcal{E}' \hookrightarrow \mathcal{E}$
such that any subspace $X' \hookrightarrow X$ defines a subtopos $\mathcal{E}_{X'} \hookrightarrow \mathcal{E}_X$.
Usual geometric operations (union, intersection, push-forward, pull-back) generalize from topological spaces to arbitrary toposes.

(4) Invariant properties (local connectedness, connectedness, $\cdots$) and topological invariants (homotopy, cohomology, $\cdots$) generalize to toposes.

Morphism of toposes

Definition. – *A morphism of toposes* $\mathcal{E}' \xrightarrow{f} \mathcal{E}$ *is a pair of adjoint functors*

$$(\mathcal{E} \xrightarrow{f^*} \mathcal{E}' , \mathcal{E}' \xrightarrow{f_*} \mathcal{E})$$

such that f^* respects not only arbitrary colimits but also finite limits.

Justification:

- Any continous map $f : X \to Y$

  defines a topos morphism $(\mathcal{E}_Y \xrightarrow{f^*} \mathcal{E}_X , \mathcal{E}_X \xrightarrow{f_*} \mathcal{E}_Y)$

$$f^* : (F = \text{ sheaf on } Y) \longmapsto f^*F = \Big(\underset{\text{open in } X}{U} \mapsto (f^*F)(U) = \varinjlim_{\substack{V \text{ open in } Y \\ U \subseteq f^{-1}V}} F(V) \Big),$$

$$f_* : (G = \text{ sheaf on } X) \longmapsto f_*G = \Big(\underset{\text{open in } Y}{V} \mapsto (f_*G)(V) = G(f^{-1}V) \Big).$$

- If Y is "sober", the map

$$\begin{array}{ccc} \{\text{continuous maps } X \to Y\} & \longrightarrow & \{\text{topos morphisms } \mathcal{E}_X \to \mathcal{E}_Y\}, \\ (X \xrightarrow{f} Y) & \longmapsto & (\mathcal{E}_Y \xrightarrow{f^*} \mathcal{E}_X , \mathcal{E}_X \xrightarrow{f_*} \mathcal{E}_Y) \end{array}$$

  is one-to-one.

Remark:

Morphisms from $\mathcal{E}'$ to $\mathcal{E}$ make up a category $\mathrm{Geom}(\mathcal{E}', \mathcal{E})$

by defining transforms of topos morphisms $(\mathcal{E}' \xrightarrow{f} \mathcal{E}) \to (\mathcal{E}' \xrightarrow{g} \mathcal{E})$

to be transforms of functors $f^* \to g^*$.

Points of toposes

Definition. –

A point of a topos $\mathcal{E}$

is a topos morphism

$$\text{Set} \longrightarrow \mathcal{E},$$

consisting in a pair of adjoint functors $(\mathcal{E} \to \text{Set}\,,\ \text{Set} \to \mathcal{E})$.

Justification:

- Any point $x \in X$ of a topological space X

  defines a point of the associated topos

  $$\text{Set} = \mathcal{E}_{\{\bullet\}} \longrightarrow \mathcal{E}$$

  consisting in a pair of adjoint functors

  $$(\mathcal{E}_X \xrightarrow{x^*} \text{Set}\,,\ \text{Set} \xrightarrow{x_*} \mathcal{E}_X).$$

- If X is "sober", the map

  $$\begin{array}{ccc} \{\text{points of } X\} & \longrightarrow & \{\text{points of the topos } \mathcal{E}_X\}, \\ (x \in X) & \longmapsto & (\mathcal{E}_X \xrightarrow{x^*} \text{Set}\,,\ \text{Set} \xrightarrow{x_*} \mathcal{E}) \end{array}$$

  is one-to-one.

Subtoposes

Definition. –

(i) *A topos morphism* $\mathcal{E}' \xrightarrow{j} \mathcal{E}$ *consisting in*

$$(\mathcal{E} \xrightarrow{j^*} \mathcal{E}' \,,\, \mathcal{E}' \xrightarrow{j_*} \mathcal{E})$$

is called an "embedding" if

$$j_* : \mathcal{E}' \longrightarrow \mathcal{E} \quad \text{is fully faithful.}$$

(ii) *A subtopos of a topos* $\mathcal{E}$ *is an equivalence class of embeddings*

$$\mathcal{E}' \hookrightarrow \mathcal{E}.$$

Justification:

Any subspace of a topological space

$$X' \hookrightarrow X$$

defines a subtopos

$$\mathcal{E}_{X'} \hookrightarrow \mathcal{E}_X.$$

Remark:

If X is a "sober" topological space, the map

$$\{\text{subspaces of } X\} \longrightarrow \{\text{subtoposes of } \mathcal{E}_X\}$$

is injective but not surjective (except in trivial cases).

There are more subtoposes than subspaces

and they are better behaved.

Subtoposes and Grothendieck topologies

Theorem. –

(i) *If J is a topology on a small or essentially small category $\mathcal{C}$, it defines a subtopos*

$$\left(\widehat{\mathcal{C}}_J \xhookrightarrow{j} \widehat{\mathcal{C}}\right) = \left(\widehat{\mathcal{C}} \xrightarrow{j^*} \widehat{\mathcal{C}}_J \, , \, \widehat{\mathcal{C}}_J \xhookrightarrow{j_*} \widehat{\mathcal{C}}\right).$$

(ii) *If $\mathcal{E} \cong \widehat{\mathcal{C}}_J$, the map*

$$\begin{array}{ccc} \{\text{topologies } J' \supseteq J \text{ on } \mathcal{C}\} & \longrightarrow & \{\text{subtoposes of } \mathcal{E}\}\,, \\ J' & \longmapsto & (\widehat{\mathcal{C}}_{J'} \hookrightarrow \widehat{\mathcal{C}}_J \cong \mathcal{E}) \end{array}$$

is one-to-one.

Corollary:

(i) *Subtoposes of a topos make up a set ordered by inclusion, with arbitrary unions and intersections.*

(ii) *Any pair of subtoposes $\mathcal{E}_1 \hookrightarrow \mathcal{E}$, $\mathcal{E}_2 \hookrightarrow \mathcal{E}$ define a difference*

$$\mathcal{E}_2 \backslash \mathcal{E}_1 \lhook\joinrel\longrightarrow \mathcal{E} \quad \textit{characterized by}$$
$$\mathcal{E}' \supseteq \mathcal{E}_2 \backslash \mathcal{E}_1 \Leftrightarrow \mathcal{E}' \cup \mathcal{E}_1 \supseteq \mathcal{E}_2 \, , \, \forall (\mathcal{E}' \hookrightarrow \mathcal{E}).$$

(iii) *Unions and intersections verify*

$$\mathcal{E}' \cup \left(\bigcap_{i \in I} \mathcal{E}_i\right) = \bigcap_{i \in I} (\mathcal{E}' \cup \mathcal{E}_i),$$
$$\mathcal{E}' \cap (\mathcal{E}_1 \cup \mathcal{E}_2) = (\mathcal{E}' \cap \mathcal{E}_1) \cup (\mathcal{E}' \cap \mathcal{E}_2).$$

Geometric transforms of subtoposes

Proposition. –

Any topos morphism $\mathcal{E}' \xrightarrow{f} \mathcal{E}$ *uniquely factorizes as*

$$\mathcal{E}' \overset{\bar{f}}{\twoheadrightarrow} \mathrm{Im}(f) \hookrightarrow \mathcal{E} \quad \textit{where}$$

$$\begin{cases} \bullet \quad \mathrm{Im}\,(f) \hookrightarrow \mathcal{E} \text{ is an embedding,} \\ \bullet \quad \mathcal{E}' \overset{\bar{f}}{\twoheadrightarrow} \mathrm{Im}(f) \text{ is “surjective” in the sense that} \\ \quad\;\; \bar{f}^* : \mathrm{Im}(f) \longrightarrow \mathcal{E}' \text{ is faithful.} \end{cases}$$

Theorem. –

Let $f : \mathcal{E}' \to \mathcal{E}$ *be a topos morphism.*

(i) *The push-forward map*

$$\begin{array}{rcccl} f_* & : & \{\text{subtoposes } \mathcal{E}'_1 \hookrightarrow \mathcal{E}'\} & \longrightarrow & \{\text{subtoposes } \mathcal{E}_1 \hookrightarrow \mathcal{E}\}, \\ & & (\mathcal{E}'_1 \hookrightarrow \mathcal{E}') & \longmapsto & f_*\mathcal{E}'_1 = \mathrm{Im}(\mathcal{E}'_1 \hookrightarrow \mathcal{E}' \xrightarrow{f} \mathcal{E}) \end{array}$$

has a left adjoint

f^{-1} = *pull-back of subtoposes* $\mathcal{E}_1 \hookrightarrow \mathcal{E}$, *characterized by*

$$f^{-1}\mathcal{E}_1 \supseteq \mathcal{E}'_1 \Leftrightarrow \mathcal{E}_1 \supseteq f_*\mathcal{E}'_1 \,.$$

(ii) *(O. C., L. L.) If* f *is “locally connected”,*

f^{-1} *also has a left-adjoint* $f_!$ *characterized by*

$$f_!\mathcal{E}'_1 \supseteq \mathcal{E}_1 \Leftrightarrow \mathcal{E}'_1 \supseteq f^{-1}\mathcal{E}_1 \,.$$

Canonical functors and representations of topos morphisms

Definition. –
For any site $(\mathcal{C}, J)$*, the associated canonical functor is*

$$\ell = j^* \circ y : \mathcal{C} \hookrightarrow \widehat{\mathcal{C}} \xrightarrow{j^*} \widehat{\mathcal{C}}_J \,.$$

Remark:
The canonical functor ℓ is fully faithful if and only if J is "subcanonical".

Theorem (Grothendieck, Diaconescu). –
If $\mathcal{E}'$ *and* $\mathcal{E} \xrightarrow{\sim} \widehat{\mathcal{C}}_J$ *are toposes, the functor*

$$\begin{array}{rcl} \mathrm{Geom}(\mathcal{E}', \mathcal{E}) & \longrightarrow & \{\textit{category of functors } \mathcal{C} \to \mathcal{E}'\}, \\ (\mathcal{E}' \xrightarrow{f} \mathcal{E}) & \longmapsto & \rho = (\mathcal{C} \xrightarrow{\ell} \widehat{\mathcal{C}}_J \xrightarrow{\sim} \mathcal{E} \xrightarrow{f^*} \mathcal{E}') \end{array}$$

is an equivalence to the full subcategory of functors

$$\rho : \mathcal{C} \longrightarrow \mathcal{E}'$$

which are

$$\begin{cases} \bullet \quad \text{“}\textit{flat}\text{”}, \\ \bullet \quad \text{“}J\textit{-continuous}\text{”, i.e. transform } J\textit{-covering families} \\ \quad\;\; (X_i \xrightarrow{x_i} X)_{i \in I} \textit{ into globally epimorphic families of } \mathcal{E}'. \end{cases}$$

Remark:
If $\mathcal{C}$ is "cartesian", i.e. has finite limits, ρ is "flat" if and only if it respects finite limits.

Logical interpretation of functors

Definition. –

If $\mathcal{C}$ is a small category, let $\Sigma_{\mathcal{C}}$ be the theory without axioms consisting in the vocabulary

$$\{\text{names of objects}\} = \{\text{list of objects } X \text{ of } \mathcal{C}\}\,,$$

$$\{\text{names of morphisms}\} = \{\text{list of arrows } X \xrightarrow{f} Y \text{ of } \mathcal{C}\}$$

so that a "model" ρ of $\Sigma_{\mathcal{C}}$ in a topos $\mathcal{E}'$ consists in the choice of

$$\begin{cases} \bullet \quad \text{an object } \rho(X) \text{ of } \mathcal{E}' \text{ for any object } X \text{ of } \mathcal{C}, \\ \bullet \quad \text{a morphism } \rho(X) \xrightarrow{\rho(f)} \rho(Y) \text{ of } \mathcal{E}' \text{ for any arrow } X \xrightarrow{f} Y \text{ of } \mathcal{C}. \end{cases}$$

Lemma. –

A "model" ρ of $\Sigma_{\mathcal{C}}$ in a topos $\mathcal{E}'$ is a functor if and only if it verifies the following axioms which define a first-order theory $\mathbb{T}_{\mathcal{C}}$ in the language $\Sigma_{\mathcal{C}}$:

$$\begin{cases} \text{(A1)} \quad \text{For any arrows } X \xrightarrow{f} Y \xrightarrow{g} Z \text{ of } \mathcal{C} \\ \qquad\qquad \top \vdash_x (g \circ f)(x) = g(f(x))\,. \\ \text{(A2)} \quad \text{For any object } X \text{ of } \mathcal{C}, \\ \qquad\qquad \top \vdash_x \mathrm{id}_X(x) = x\,. \end{cases}$$

Logical interpretation of flatness and continuity

Proposition. –

(i) *A "model" ρ of $\mathbb{T}_{\mathcal{C}}$ in a topos $\mathcal{E}'$, i.e. a functor* $\rho : \mathcal{C} \longrightarrow \mathcal{E}'$
is "flat" if and only if it verifies the following axioms wich define a "quotient theory" $\mathbb{T}^f_{\mathcal{C}}$ of $\mathbb{T}_{\mathcal{C}}$:

$$\begin{cases} \text{(B1)} \quad \top \vdash \bigvee\limits_{X \in \mathrm{Ob}(\mathcal{C})} (\exists\, x^X)\, \top(x^X)\,. \\[2ex] \text{(B2)} \quad \textit{For any pair of objects } X, Y \textit{ of } \mathcal{C} \\ \qquad \top \vdash_{x^X, y^Y} \bigvee\limits_{\substack{W \in \mathrm{Ob}(\mathcal{C}) \\ (f,g) \in \mathrm{Hom}(W,X) \times \mathrm{Hom}(W,Y)}} (\exists\, w^W)(x^X = f(w^W) \wedge y^Y = g(w^W))\,. \\[2ex] \text{(B3)} \quad \textit{For any pair of arrows } X \underset{g}{\overset{f}{\rightrightarrows}} Y \textit{ of } \mathcal{C} \\ \qquad f(x^X) = g(x^X) \vdash_{x^X} \bigvee\limits_{\substack{W \in \mathrm{Ob}\,\mathcal{C} \\ h \in \mathrm{Hom}(W,X) \\ \textit{such that } f \circ h = g \circ h}} (\exists\, w^W)(x^X = h(w^W))\,. \end{cases}$$

(ii) *A "model" ρ of $\mathbb{T}^f_{\mathcal{C}}$ in a topos $\mathcal{E}'$, i.e. a flat functor* $\rho : \mathcal{C} \longrightarrow \mathcal{E}'$
is "J-continuous" if and only if it verifies the following axioms
wich define a "quotient theory" $\mathbb{T}^f_{\mathcal{C},J}$ of $\mathbb{T}^f_{\mathcal{C}}$:

$$\begin{cases} \text{(C)} \quad \textit{For any J-covering presieve } (X_i \xrightarrow{f_i} X)_{i \in I} \textit{ of } \mathcal{C}, \\ \qquad \top(x^X) \vdash_{x^X} \bigvee\limits_{i \in I} (\exists\, x_i^{X_i})(x^X = f_i(x_i^{X_i}))\,. \end{cases}$$

First-order "geometric" theories

Definition. –

(i) *A <u>first-order language</u> Σ consists in*

$$\begin{cases} \bullet \quad \text{a family of ``\underline{names of objects}'' } A, \\ \bullet \quad \text{a family of ``\underline{names of morphisms}'' } f : A_1 \cdots A_n \longrightarrow B, \\ \bullet \quad \text{a family of ``\underline{names of subobjects}'' } R \rightarrowtail A_1 \cdots A_n \end{cases}$$

whose <u>interpretations</u> as "<u>models</u>" M in a topos $\mathcal{E}$ will consist in

$$\begin{cases} \bullet \quad \underline{\text{objects}}\ MA \text{ of } \mathcal{E} \text{ indexed by } \underline{\text{objects names}}\ A, \\ \bullet \quad \underline{\text{morphisms}}\ Mf : MA_1 \times \cdots \times MA_n \longrightarrow MB \text{ of } \mathcal{E} \text{ indexed by } \underline{\text{names}}\ f : A_1 \cdots A_n \longrightarrow B, \\ \bullet \quad \underline{\text{subobjects}}\ MR \hookrightarrow MA_1 \times \cdots \times MA_n \text{ indexed by } \underline{\text{names}}\ MR \hookrightarrow MA_1 \times \cdots \times MA_n. \end{cases}$$

(ii) *A <u>first-order "geometric" theory</u> $\mathbb{T}$ in such a <u>language</u> Σ is defined by a <u>family of axioms</u> consisting in <u>implications</u>*

$$\varphi(x_1^{A_1}, \cdots, x_n^{A_n}) \vdash \psi(x_1^{A_1}, \cdots, x_n^{A_n})$$

between <u>"geometric" formulas</u> which may be

$$\begin{cases} \bullet \quad \top \text{ or } \bot \text{ (``\underline{truth}'' or ``\underline{falsity}''),} \\ \bullet \quad \text{deduced from \underline{formulas}} \\ \qquad\qquad f(x_1^{A_1}, \cdots, x_n^{A_n}) = x^B \quad \text{or} \quad R(x_1^{A_1}, \cdots, x_n^{A_n}) \\ \quad\ \text{by \underline{substitution} or \underline{application} of the \underline{logical symbols}} \\ \quad\ \exists \text{ in part of the variables,} \quad \bigvee \text{ arbitrary disjunction,} \quad \bigwedge \text{ finite conjunction.} \end{cases}$$

Semantics of first-order geometric theories

Proposition. –

(i) *If Σ is a first-order language without axioms, the models of Σ in a topos $\mathcal{E}$ make up a category*

$$\Sigma\text{-mod}(\mathcal{E}).$$

Furthermore, any topos morphism $\mathcal{E}' \xrightarrow{f} \mathcal{E}$ induces adjoint functors

$$\big(\Sigma\text{-mod}(\mathcal{E}) \xrightarrow{f^*} \Sigma\text{-mod}(\mathcal{E}')\,,\ \Sigma\text{-mod}(\mathcal{E}') \xrightarrow{f_*} \Sigma\text{-mod}(\mathcal{E})\big).$$

(ii) *If $\mathbb{T}$ is a first-order geometric theory in a language Σ, the models of $\mathbb{T}$ in a topos $\mathcal{E}$, i.e. the models of Σ which respect the axioms of $\mathbb{T}$ make up a full subcategory*

$$\mathbb{T}\text{-mod}(\mathcal{E}) \hookrightarrow \Sigma\text{-mod}(\mathcal{E}).$$

Furthermore, any topos morphism $\mathcal{E}' \xrightarrow{f} \mathcal{E}$ induces a pull-back functor

$$\mathbb{T}\text{-mod}(\mathcal{E}) \xrightarrow{f^*} \mathbb{T}\text{-mod}(\mathcal{E}').$$

Remarks:

(i) Any pull-back component $f^* : \mathcal{E} \to \mathcal{E}'$ of a topos morphism induces a functor on models because the interpretations in toposes of symbols $\top$, $\bot$, $\exists$, $\bigvee$, $\wedge$ only use colimits and finite limits.
It would not work for symbols $\bigwedge$ (infinite conjunction), $\forall$, $\neg$ (negation).

(ii) If the interpretation of the axioms of $\mathbb{T}$ only uses limits, there is a right adjoint $\mathbb{T}\text{-mod}(\mathcal{E}') \xrightarrow{f_*} \mathbb{T}\text{-mod}(\mathcal{E})$.

The notion of classifying topos

Definition. –

Let $\mathbb{T}$ be a first-order geometric theory in a language Σ.
A topos $\mathcal{E}_{\mathbb{T}}$ is called a "classifying topos" of $\mathbb{T}$
if there exists a model $U_{\mathbb{T}}$ of $\mathbb{T}$ in $\mathcal{E}_{\mathbb{T}}$, called the "universal model",
such that, for any topos $\mathcal{E}$, the functor

$$\begin{array}{rcl} \mathrm{Geom}(\mathcal{E}, \mathcal{E}_{\mathbb{T}}) & \longrightarrow & \mathbb{T}\text{-mod}(\mathcal{E}) \,, \\ (\mathcal{E} \xrightarrow{f} \mathcal{E}_{\mathbb{T}}) & \longmapsto & f^* U_{\mathbb{T}} \end{array}$$

is an equivalence of categories.

Remark:
If a theory $\mathbb{T}$ has a "classifying topos" $\mathcal{E}_{\mathbb{T}}$,
it is unique up to equivalence.

Corollary:
Any presentation of a topos as a topos of sheaves

$$\mathcal{E} \xrightarrow{\sim} \widehat{\mathcal{C}}_J \quad \textit{on a small site } (\mathcal{C}, J)$$

yields an interpretation of $\mathcal{E}$ as a classifying topos

$$\mathcal{E} = \mathcal{E}_{\mathbb{T}^f_{\mathcal{C},J}}$$

of the "geometric theory" $\mathbb{T}^f_{\mathcal{C},J}$ of flat J-continuous functors on $\mathcal{C}$.

Classifying toposes for first-order geometric theories

The following theorem was proved in the mid 1970's in a book of Makkai and Reyes, following work of M. Hakim (as a student of Grothendieck), Joyal and others.

Theorem. –
Let $\mathbb{T}$ be a first-order geometric theory $\mathbb{T}$ in a language Σ.
Then $\mathbb{T}$ has a classifying topos

$$\mathcal{E}_{\mathbb{T}} \quad .$$

It can be constructed as a topos of sheaves

$$\mathcal{E}_{\mathbb{T}} = \widehat{(\mathcal{C}_{\mathbb{T}})}_{J_{\mathbb{T}}}$$

on a site $(\mathcal{C}_{\mathbb{T}}, J_{\mathbb{T}})$ where

- *objects of $\mathcal{C}_{\mathbb{T}}$ are geometric formulas $\varphi(x_1^{A_1}, \cdots, x_n^{A_n})$ in Σ,*
- *morphisms of $\mathcal{C}_{\mathbb{T}}$ are geometric formulas*

$$\varphi(x_1^{A_1}, \cdots, x_n^{A_n}) \xrightarrow{\theta(x_1^{A_1}, \cdots, x_n^{A_n}; y_1^{B_1}, \cdots, y_m^{B_m})} \psi(y_1^{B_1}, \cdots, y_m^{B_m})$$

  which are "$\mathbb{T}$-provably functional",
- *$J_{\mathbb{T}}$ is a subcanonical topology on $\mathcal{C}_{\mathbb{T}}$ so that the functor*
  $\ell : \mathcal{C}_{\mathbb{T}} \hookrightarrow \mathcal{E}_{\mathbb{T}}$ *is fully faithful.*

Remark:
A geometric implication $\varphi(x_1^{A_1}, \cdots, x_n^{A_n}) \vdash \psi(x_1^{A_1}, \cdots, x_n^{A_n})$ is $\mathbb{T}$-provable if and only if it is verified by the universal model $U_{\mathbb{T}}$ of $\mathbb{T}$ in $\mathcal{E}_{\mathbb{T}}$.

A duality theorem between syntax and semantics

Theorem (O. C.). –

Let $\mathbb{T}$ be a first-order geometric theory in a language Σ.
Then:

(i) *For any quotient theory $\mathbb{T}'$ of $\mathbb{T}$,*
defined in the same language Σ by adding extra axioms to those of $\mathbb{T}$,
the natural morphism between classifying toposes

$$\mathcal{E}_{\mathbb{T}'} \longrightarrow \mathcal{E}_{\mathbb{T}}$$

is an embedding.

(ii) *If quotient theories $\mathbb{T}_1$ and $\mathbb{T}_2$ of $\mathbb{T}$ are provably equivalent,*
the associated embeddings

$$\mathcal{E}_{\mathbb{T}_1} \hookrightarrow \mathcal{E}_{\mathbb{T}} \quad \textit{and} \quad \mathcal{E}_{\mathbb{T}_2} \hookrightarrow \mathcal{E}_{\mathbb{T}}$$

are equivalent.

(iii) *The map*

$$\begin{array}{ccc} \left\{ \begin{array}{l} \textit{equivalence classes of} \\ \textit{quotient theories of } \mathbb{T} \end{array} \right\} & \longrightarrow & \left\{ \textit{subtoposes of } \mathcal{E}_{\mathbb{T}} \right\}, \\ \mathbb{T}' & \longmapsto & (\mathcal{E}_{\mathbb{T}'} \hookrightarrow \mathcal{E}_{\mathbb{T}}) \end{array}$$

is one-to-one.

Consequence:
Any provability problem in first-order geometric logic
can be constructively transformed into problems of generation of Grothendieck topologies.

Proposition. –
Let E be an object of a topos $\mathcal{E}$.
Then:

(i) *The localized category $\mathcal{E}/E$ defined as*

$$\begin{cases} \bullet \quad \text{objects} = \text{morphisms } E' \to E \text{ in } \mathcal{E}, \\ \bullet \quad \text{morphisms } (E'_1 \to E) \longrightarrow (E'_2 \to E) \\ \quad\; = \text{commutative triangles} \begin{pmatrix} E'_1 & \to & E'_2 \\ & \searrow \quad \swarrow & \\ & E & \end{pmatrix} \text{ in } \mathcal{E}, \end{cases}$$

is a topos.
It is endowed with a canonical topos morphism

$$p : \mathcal{E}/E \longrightarrow \mathcal{E}$$

whose pull-back component is

$$\begin{array}{cccccc} p^* & : & \mathcal{E} & \longrightarrow & \mathcal{E}/E\,, \\ & & E' & \longmapsto & E' \times E\,. \end{array}$$

(ii) *If $\mathcal{E} = \widehat{\mathcal{C}}_J$ is the topos of sheaves on a site $(\mathcal{C}, J)$,*

$$\mathcal{E}/E \quad \text{identifies with} \quad \widehat{(\mathcal{C}/E)}_J$$

where $\mathcal{C}/E$ is the "category of elements" of E defined as

$$\begin{cases} \bullet \quad \text{objects} = \text{pairs } (X, e) \text{ with } X \in \mathrm{Ob}(\mathcal{C}),\, e \in E(X), \\ \bullet \quad \text{morphisms } (X', e') \to (X, e) \\ \quad\; = \text{morphisms } X' \xrightarrow{x} X \text{ such that } E(x)(e) = e'. \end{cases}$$

Localized toposes as solutions of classification problems

Interpretation. –

Consider a classification problem

$$(\mathcal{C}, M_{\mathbb{T}} : \mathcal{C}^{\mathrm{op}} \longrightarrow \mathrm{Set})$$

consisting in a presheaf on a small or essentially small category

$$M_{\mathbb{T}} : \mathcal{C}^{\mathrm{op}} \longrightarrow \mathrm{Set}$$

which is a sheaf for a topology J on $\mathcal{C}$.

Then Giraud (as a student of Grothendieck)

calls the associated "relative topos"

$$\begin{array}{ccc} \widehat{\mathcal{C}}_J / M_{\mathbb{T}} & \cong & (\widehat{\mathcal{C}/M_{\mathbb{T}}})_J \\ \downarrow & & \\ \widehat{\mathcal{C}}_J & & \end{array}$$

a "classifying topos" for the classification problem $M_{\mathbb{T}}$.

Question: Why?

The answer is part of the theory of relative toposes $\cdots$

Definition. –

(i) *A category $\mathcal{X}$ can be called "topological"*
if it is endowed with a functor to the category of toposes

$$\begin{cases} \mathcal{X} & \longrightarrow & \mathrm{Top} = \textit{category of toposes}, \\ X & \longmapsto & \mathcal{E}_X\,, \\ (X \xrightarrow{f} Y) & \longmapsto & (\mathcal{E}_X \xrightarrow{(f^*, f_*)} \mathcal{E}_Y). \end{cases}$$

(ii) *A property of objects X of such a topological category $\mathcal{X}$*
can be called "topological" if it only depends on the associated topos

$$\mathcal{E}_X \qquad \textit{considered up to equivalence.}$$

(iii) *A topological invariant of such a category $\mathcal{X}$ is a functor*
to a category $\mathcal{A}$ (usually algebraic)

$$\mathcal{X} \longrightarrow \mathcal{A} \quad \textit{or} \quad \mathcal{X}^{\mathrm{op}} \longrightarrow \mathcal{A}$$

which factorizes through the functor

$$\begin{array}{ccc} \mathcal{X} & \longrightarrow & \mathrm{Top}\,, \\ X & \longmapsto & \mathcal{E}_X\,. \end{array}$$

Examples of topological categories

Example: the category of topological spaces X with

$$X \longmapsto \mathcal{E}_X = \text{topos of sheaves on } X.$$

Example: the category of simplicial sets $X_\bullet$ with

$$X_\bullet \longmapsto |X_\bullet| (= \text{ topological realization}) \longmapsto \mathcal{E}_{|X_\bullet|}.$$

Example: the category of small categories $\mathcal{C}$ with

$$\mathcal{C} \mapsto \widehat{\mathcal{C}}, \quad (\mathcal{D} \xrightarrow{\rho} \mathcal{C}) \mapsto (\widehat{\mathcal{C}} \xrightarrow{\rho^* = \rho \circ (\bullet)} \widehat{\mathcal{D}},\ \widehat{\mathcal{D}} \xrightarrow{\rho_*} \widehat{\mathcal{C}}).$$

Example: the category of small sites $(\mathcal{C}, J)$
and comorphisms [resp. morphisms] of sites

$$(\mathcal{D}, K) \longrightarrow (\mathcal{C}, J)$$

consisting in functors

$$\mathcal{D} \xrightarrow{\rho} \mathcal{C} \qquad [\text{resp. } \mathcal{C} \xrightarrow{\rho} \mathcal{D}]$$

which induce a topos morphism

$$\widehat{\mathcal{D}}_K \xrightarrow{(\rho^*, \rho_*)} \widehat{\mathcal{C}}_J \qquad [\text{resp. } \widehat{\mathcal{D}}_K \xrightarrow{(\rho_!, \rho^!)} \widehat{\mathcal{C}}_J]$$

in the sense that there is a commutative square

$$\begin{tikzcd} \widehat{\mathcal{C}} \arrow[r, "\rho \circ (\bullet)"] \arrow[d, "j^*"'] & \widehat{\mathcal{D}} \arrow[d, "j^*"] \\ \widehat{\mathcal{C}}_J \arrow[r, "\rho^*"] & \widehat{\mathcal{D}}_K \end{tikzcd} \qquad [\text{resp.} \quad \begin{tikzcd} \mathcal{C} \arrow[r, "\rho"] \arrow[d, "\ell"'] & \mathcal{D} \arrow[d, "\ell"] \\ \widehat{\mathcal{C}}_J \arrow[r, "\rho_!"] & \widehat{\mathcal{D}}_K \end{tikzcd} \quad].$$

The topological category of first-order geometric theories

Recall that any first-order geometric theory $\mathbb{T}$ has a classifying topos

$$\mathcal{E}_{\mathbb{T}}$$

which may be constructed as the topos of sheaves

$$\widehat{(\mathcal{C}_{\mathbb{T}})}_{J_{\mathbb{T}}}$$

on the syntactic category $\mathcal{C}_{\mathbb{T}}$ of $\mathbb{T}$ (consisting in geometric formulas)
endowed with the syntactic topology $J_{\mathbb{T}}$.

Definition. –

First-order geometric theories $\mathbb{T}$ make up a topological category
by defining morphisms

$$\mathbb{T}' \longrightarrow \mathbb{T}$$

as functors between syntactic categories

$$\mathcal{C}_{\mathbb{T}} \longrightarrow \mathcal{C}_{\mathbb{T}'}$$

which

$$\begin{cases} \bullet & \textit{respect finite limits,} \\ \bullet & \textit{transform } J_{\mathbb{T}}\textit{-covering families into } J_{\mathbb{T}'}\textit{-covering families,} \end{cases}$$

so as to induce a morphism of classifying toposes

$$\mathcal{E}_{\mathbb{T}'} \longrightarrow \mathcal{E}_{\mathbb{T}}.$$

A principle of the theory of "toposes as bridges"

- Consider a topological category

$$\begin{array}{ccl} \mathcal{X} & \longrightarrow & \mathrm{Top} = \{\text{category of toposes}\}, \\ X & \longmapsto & \mathcal{E}_X \,. \end{array}$$

- Consider a "topological property" of objects X of $\mathcal{X}$, depending only on $\mathcal{E}_X$ considered up to equivalence.

- Or consider a "topological invariant", i.e. a functor

$$\mathcal{X} \longrightarrow \mathcal{A} \qquad \text{or} \qquad \mathcal{X}^{\mathrm{op}} \longrightarrow \mathcal{A}$$

which factorizes through

$$\begin{array}{ccl} \mathcal{X} & \longrightarrow & \mathrm{Top}\,, \\ X & \longmapsto & \mathcal{E}_X \,. \end{array}$$

- Then express or compute concretely

this "topological property"

or

this "topological invariant"

in terms of the objects X of $\mathcal{X}$.

A classical topological invariant: Poincaré groupoids

Reminder:

A groupoid is a category whose arrows are all invertible.

In particular, the endomorphisms of any object X make up a group $\mathrm{Aut}(X)$.

Furthermore, any morphism $X' \xrightarrow{\sim} X$ induces an isomorphism $\mathrm{Aut}(X') \xrightarrow{\sim} \mathrm{Aut}(X)$.

Definition. –

The Poincaré groupoid π_X of a topological space X consists in

$$\begin{cases} \bullet \quad \textit{objects } x \textit{ which are the elements } x \in X, \\ \bullet \quad \textit{invertible arrows } x' \longrightarrow x \textit{ which are} \\ \quad\ \textit{the homotopy classes of continous paths } \begin{cases} \sigma : [0,1] \to X \\ \textit{such that } \sigma(0) = x'\,,\ \sigma(1) = x\,. \end{cases} \end{cases}$$

Observation. –

The construction of Poincaré groupoids defines a functor

$$\begin{array}{rcl} \{\textit{category of topological spaces}\} & \longrightarrow & \{\textit{category of small groupoids}\}, \\ X & \longmapsto & \pi_X\,. \end{array}$$

Étale covers of topological spaces

Definition. –
Let X be a topological space.

(i) *A locally constant étale cover of X is a space over X*

$$X' \xrightarrow{p} X$$

such that there exist

- *an open covering of X by open subsets*
$$U_i \subset X\,, \quad i \in I\,,$$
- *homeomorphisms of the restrictions $p^{-1}(U_i) = X' \times_X U_i$*
$$X' \times_X U_i \xrightarrow{\sim} \coprod_{I_i} U_i \quad \text{over } U_i, \text{ for } i \in I\,,$$
- *open coverings of the intersections $U_{i_1} \cap U_{i_2}$, $i_1, i_2 \in I$*
by open subsets $U_{i_1,i_2,j} \subset U_{i_1} \cap U_{i_2}$
verifying the condition that the composed homeomorphism
$$\coprod_{I_{i_1}} U_{i_1,i_2,j} \xrightarrow{\sim} X' \times_X U_{i_1,i_2,j} \xrightarrow{\sim} \coprod_{I_{i_2}} U_{i_1,i_2,j}$$
is defined by a bijection $I_{i_1} \xrightarrow{\sim} I_{i_2}$.

(ii) *An étale cover of X is a space over X*

$$X' \xrightarrow{p} X$$

which can be written as a disjoint sum of locally constant étale covers.

Grothendieck-Poincaré groupoids of topological spaces

Proposition. –

(i) *For any topological space X, the category of étale covers of X*
$$\mathrm{Et}_X \quad \textit{is a topos.}$$
Furthermore, the category of points of this topos
$$\pi'_X = \mathrm{Geom}(\mathrm{Set}, \mathrm{Et}_X) \quad \textit{is a groupoid.}$$

(ii) *Any continuous map $X \xrightarrow{f} Y$ induces a topos morphism*
$$(f^*, f_*) : \mathrm{Et}_X \longrightarrow \mathrm{Et}_Y$$
whose pull-back component f^* is
$$(Y' \to Y) \longmapsto (Y' \times_Y X \to X)\,,$$
and this topos morphism yields a functor
$$\pi'_X \longrightarrow \pi'_Y\,.$$

Theorem. –

(i) *There is a natural transformation from the Poincaré functor*
$$\pi_\bullet : \{\textit{topological spaces}\} \longrightarrow \{\textit{groupoids}\}$$
to the Grothendieck-Poincaré functor
$$\pi'_\bullet : \{\textit{topological spaces}\} \longrightarrow \{\textit{groupoids}\}.$$

(ii) *For any topological space X which is locally simply connected, the associated functor*
$$\pi_X \longrightarrow \pi'_X$$
is an equivalence of groupoids.

Étale covers in toposes

Definition. –
Let $\mathcal{E}$ be a topos and $1_{\mathcal{E}}$ be its terminal object.

(i) *An object E of $\mathcal{E}$ is called "locally constant"*

if there exist

- *a globally epimorphic family*

$$U_i \longrightarrow 1_{\mathcal{E}}\,, \quad i \in I\,,$$

- *isomorphisms*

$$E \times U_i \xrightarrow{\sim} \coprod_{I_i} U_i \quad \text{over } U_i \text{ for any } i \in I,$$

- *globally epimorphic families on all products $U_{i_1} \times U_{i_2}$, $i_1, i_2 \in I$,*

$$U_{i_1,i_2,j} \longrightarrow U_{i_1} \times U_{i_2}$$

verifying the condition that the composed isomorphism

$$\coprod_{I_{i_1}} U_{i_1,i_2,j} \xrightarrow{\sim} E \times U_{i_1,i_2,j} \xrightarrow{\sim} \coprod_{I_{i_2}} U_{i_1,i_2,j}$$

is defined by a bijection $I_{i_1} \xrightarrow{\sim} I_{i_2}$.

(ii) *An object E of $\mathcal{E}$ is called an "étale cover"*

if it can be written as a disjoint sum of locally constant objects of $\mathcal{E}$.

Grothendieck groupoids of toposes

Proposition. –

(i) *For any topos $\mathcal{E}$, the category of étale covers in $\mathcal{E}$*

$$\mathrm{Et}_{\mathcal{E}} \qquad \textit{is a topos.}$$

Furthermore, the category of points of this topos

$$\pi'_{\mathcal{E}} = \mathrm{Geom}(\mathrm{Set}, \mathrm{Et}_{\mathcal{E}}) \qquad \textit{is a groupoid.}$$

(ii) *Any topos morphism $\mathcal{E}' \xrightarrow{f} \mathcal{E}$ induces a topos morphism*

$$\mathrm{Et}_{\mathcal{E}'} \longrightarrow \mathrm{Et}_{\mathcal{E}}$$

whose pull-back component

$$\mathrm{Et}_{\mathcal{E}} \xrightarrow{f^*} \mathrm{Et}_{\mathcal{E}'}$$

is the restriction of

$$\mathcal{E} \xrightarrow{f^*} \mathcal{E}'.$$

Proposition. –
The Grothendieck-Poincaré functor

$$\pi'_{\bullet} : \{\textit{topological spaces}\} \longrightarrow \{\textit{groupoids}\}$$

factorizes as the composite

$$\{\textit{topological spaces}\} \xrightarrow{X \mapsto \mathcal{E}_X} \{\textit{toposes}\} \xrightarrow{\pi'_{\bullet}} \{\textit{groupoids}\}.$$

Indeed, for any topological space X, the toposes

$$\mathrm{Et}_X \quad \textit{and} \quad \mathrm{Et}_{\mathcal{E}_X} \quad \textit{identify.}$$

Geometric categories and geometric invariants

Definition. –
Let $\mathbb{T}$ be a first-order geometric theory.

(i) *A category $\mathcal{X}$ can be called "geometric of type $\mathbb{T}$" if it is endowed with a functor*

$$\begin{cases} \mathcal{X} & \longrightarrow & \mathrm{Top}_{\mathbb{T}} = \text{ category of pairs } (\mathcal{E} = \text{topos}, \mathcal{O} = \text{model of } \mathbb{T} \text{ in } \mathcal{E}), \\ X & \longmapsto & (\mathcal{E}_X, \mathcal{O}_X)\,, \\ (X \xrightarrow{f} Y) & \longmapsto & (\mathcal{E}_X \xrightarrow{(f^*, f_*)} \mathcal{E}_Y\,,\, f^*\mathcal{O}_Y \to \mathcal{O}_X). \end{cases}$$

(ii) *A property of objects X of such a geometric category $\mathcal{X}$ can be called "geometric" if it only depends on the associated pair*

$$(\mathcal{E}_X, \mathcal{O}_X) \quad \text{considered up to equivalence.}$$

(iii) *A geometric invariant of such a category $\mathcal{X}$ is a functor to a category $\mathcal{A}$ (usually algebraic)*

$$\mathcal{X} \longrightarrow \mathcal{A} \quad \text{or} \quad \mathcal{X}^{\mathrm{op}} \longrightarrow \mathcal{A}$$

which factorizes through the functor

$$\begin{array}{ccc} \mathcal{X} & \longrightarrow & \mathrm{Top}_{\mathbb{T}}\,, \\ X & \longmapsto & (\mathcal{E}_X, \mathcal{O}_X)\,. \end{array}$$

Remark: If the axioms of $\mathbb{T}$ interpret only in terms of finite limits,
a morphism of $\mathbb{T}$-models in $\mathcal{E}_X$ $\quad f^*\mathcal{O}_Y \longrightarrow \mathcal{O}_X$
amounts to a morphism of $\mathbb{T}$-models in $\mathcal{E}_Y$ $\quad \mathcal{O}_Y \longrightarrow f_*\mathcal{O}_Y\,.$

Examples of geometric categories of "local rings" type

Notations:

Let $\mathbb{T}_r$ = algebraic theory of rings,
$\mathbb{T}_{cr}$ = quotient theory of commutative rings,
$\mathbb{T}_{\ell cr}$ = quotient theory of local commutative rings,
defined by the extra axiom

$$(\exists z)(z \cdot (x+y) = 1) \vdash_{x,y} (\exists x')(x' \cdot x = 1) \vee (\exists y')(y' \cdot y = 1)\,.$$

Examples of geometric categories of type $\mathbb{T}_{\ell cr}$:

- Diff = category of differential manifolds

$$X \longmapsto (\mathcal{E}_X, \mathcal{O}_X = \text{sheaf of } \infty\text{-differentiable functions}).$$

- An = category of analytic varieties

$$X \longmapsto (\mathcal{E}_X, \mathcal{O}_X = \text{sheaf of holomorphic functions}).$$

- Sch = category of schemes

$$\begin{aligned} X &\longmapsto (\mathcal{E}_X = \text{Zariski topos}, \mathcal{O}_X = \text{structure sheaf}), \\ X &\longmapsto (\mathcal{E}_X^{\text{ét}} = \text{étale topos}, \mathcal{O}_X = \text{constant sheaf } \mathbb{Z}/\ell^n\mathbb{Z} \text{ for } \ell = \text{prime}), \\ X &\longmapsto (\mathcal{E}_X^{\text{pro}} = \text{pro-étale topos}, \mathcal{O}_X = \text{constant sheaf } \mathbb{Z}_\ell). \end{aligned}$$

Examples of geometric categories of "rings" type

Notation:

Let

$\mathrm{Diff}_n =$ category of n-dim. differential manifolds
and of ∞-differentiable maps

$$X' \longrightarrow X$$

which are locally invertible.

Observation:

This category is geometrized by the functor

$$X \longmapsto (\mathcal{E}_X, \mathcal{D}_X)$$

$\|$

sheaf of (non-commutative) rings

of linear differential operators,

generated over $\mathcal{O}_X$ by

$$\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_n}$$

if $(x_1, x_2, \cdots, x_n) =$ local coordinates.

Categories of inner modules over inner rings of toposes

Definition. – *Consider a geometric category of "rings" type*

$$\begin{array}{rcl} \mathcal{X} & \longrightarrow & \mathrm{Top}_{\mathbb{T}_r}\,, \\ X & \longmapsto & (\mathcal{E}_X, \mathcal{R}_X)\,, \\ (X \xrightarrow{f} Y) & \longmapsto & (\mathcal{E}_X \xrightarrow{(f^*, f_*)} \mathcal{E}_Y\,,\ f^*\mathcal{R}_Y \to \mathcal{R}_X). \\ & & \qquad\qquad\qquad\qquad \Updownarrow \\ & & \qquad\qquad\qquad\qquad \mathcal{R}_Y \to f_*\mathcal{R}_X \end{array}$$

Then one associates

- *to any object X of $\mathcal{X}$ the abelian category*

  $\mathcal{M}od_{\mathcal{R}_X}$ *of inner modules over $\mathcal{R}_X$ in $\mathcal{E}_X$* ,

- *to any morphism $X \xrightarrow{f} Y$ of $\mathcal{X}$ the linear functor*

  $$f_* : \mathcal{M}od_{\mathcal{R}_X} \longrightarrow \mathcal{M}od_{\mathcal{R}_Y}$$

  induced by $f_* : \mathcal{E}_X \to \mathcal{E}_Y$ and $\mathcal{R}_Y \to f_*\mathcal{R}_X$,

  and its left-adjoint

  $$f^* : \mathcal{M}od_{\mathcal{R}_Y} \longrightarrow \mathcal{M}od_{f^*\mathcal{R}_Y} \xrightarrow{\mathcal{R}_X \otimes_{f^*\mathcal{R}_Y} \bullet} \mathcal{M}od_{\mathcal{R}_X}$$

  induced by $f^* : \mathcal{E}_Y \to \mathcal{E}_X$ and $f^*\mathcal{R}_Y \to \mathcal{R}_X$.

Remark: If $\mathcal{E} = \widehat{\mathcal{C}}_J$ is a topos of sheaves on a site $(\mathcal{C}, J)$,
an inner ring $\mathcal{R}$ of $\mathcal{E}$ is just a J-sheaf of rings on $\mathcal{C}$
and an inner module over $\mathcal{R}$ in $\mathcal{E}$ is just a J-sheaf of modules.

Global section functors as topological or geometric invariants

Observation. –
Consider a category $\mathcal{X}$ with a terminal object X_0,
meaning any object X has a unique morphism $p_X : X \to X_0$.

(i) *If $\mathcal{X}$ is a "topological category", i.e. is endowed with*

$$\begin{array}{ccl} \mathcal{X} & \longrightarrow & \mathrm{Top} = \textit{category of toposes}, \\ X & \longmapsto & \mathcal{E}_X\,, \end{array}$$

the construction $X \mapsto (p_X)_* \circ p_X^* : \mathcal{E}_{X_0} \to \mathcal{E}_{X_0}$
defines a topological invariant

$$\mathcal{X}^{\mathrm{op}} \longrightarrow \{\textit{category of functors } \mathcal{E}_{X_0} \to \mathcal{E}_{X_0}\}.$$

(ii) *If $\mathcal{X}$ is a "geometric category" of "rings" type, endowed with*

$$\begin{array}{ccl} \mathcal{X} & \longrightarrow & \mathrm{Top}_{\mathbb{T}_r}\,, \\ X & \longmapsto & (\mathcal{E}_X, \mathcal{R}_X)\,, \end{array}$$

the construction $X \mapsto [(p_X)_* \circ p_X^* : \mathcal{M}od_{\mathcal{R}_{X_0}} \to \mathcal{M}od_{\mathcal{R}_{X_0}}]$
defines a geometric invariant

$$\mathcal{X}^{\mathrm{op}} \longrightarrow \{\textit{category of } \mathcal{R}_{X_0}\textit{-linear functors } \mathcal{M}od_{\mathcal{R}_{X_0}} \to \mathcal{M}od_{\mathcal{R}_{X_0}}\}.$$

Remark:
Indeed, any morphism $X \xrightarrow{f} Y$ in $\mathcal{X}$ yields a natural transformation

$$(p_Y)_* \circ p_Y^* \longrightarrow (p_X)_* \circ p_X^* \quad \text{induced by } \mathrm{Id} \to f_* \circ f^*.$$

Natural operations on modules over a commutative ring in a topos

Proposition. –

Consider a topos $\mathcal{E}$ endowed with an inner commutative ring $\mathcal{O}$,

and the associated abelian category $\mathcal{M}od_{\mathcal{O}}$ of inner modules over $\mathcal{O}$ in $\mathcal{E}$.

(i) *There is a well-defined "tensor product" functor*

$$\begin{array}{rcl} \mathcal{M}od_{\mathcal{O}} \times \mathcal{M}od_{\mathcal{O}} & \longrightarrow & \mathcal{M}od_{\mathcal{O}}\,, \\ (\mathcal{M}_1, \mathcal{M}_2) & \longmapsto & \mathcal{M}_1 \otimes_{\mathcal{O}} \mathcal{M}_2 \end{array}$$

characterized by the property that, for any inner module $\mathcal{M}$,

$\mathcal{O}$-linear morphisms $\qquad \mathcal{M}_1 \otimes_{\mathcal{O}} \mathcal{M}_2 \to \mathcal{M}$

identify with $\mathcal{O}$-bilinear morphisms $\qquad \mathcal{M}_1 \times \mathcal{M}_2 \to \mathcal{M}$.

(ii) *There is a well-defined "linear functionals" functor*

$$\begin{array}{rcl} \mathcal{M}od_{\mathcal{O}}^{\mathrm{op}} \times \mathcal{M}od_{\mathcal{O}} & \longrightarrow & \mathcal{M}od_{\mathcal{O}}\,, \\ (\mathcal{M}_1, \mathcal{M}_2) & \longmapsto & \mathcal{H}om_{\mathcal{O}}(\mathcal{M}_1, \mathcal{M}_2) \end{array}$$

characterized by the property that, for any inner module $\mathcal{M}$,

$\mathcal{O}$-linear morphisms $\qquad \mathcal{M} \to \mathcal{H}om_{\mathcal{O}}(\mathcal{M}_1, \mathcal{M}_2)$

identify with $\mathcal{O}$-linear morphisms $\qquad \mathcal{M} \otimes_{\mathcal{O}} \mathcal{M}_1 \to \mathcal{M}_2$.

Complexes in abelian categories

Definition. – *Let $\mathcal{A}$ be an abelian category.*

(i) *A complex in $\mathcal{A}$ is a diagram $A_\bullet$ indexed by integers $n \in \mathbb{Z}$*

$$\cdots \longrightarrow A_{n-1} \xrightarrow{d_{n-1}} A_n \xrightarrow{d_n} A_{n+1} \longrightarrow \cdots$$

such that

$$d_n \circ d_{n-1} = 0\,, \quad \forall\, n \in \mathbb{Z}\,,$$

allowing to define its "cohomology objects" in $\mathcal{A}$

$$H^n(A_\bullet) = \mathrm{Ker}(A_n \xrightarrow{d_n} A_{n+1})/\mathrm{Im}(A_{n-1} \xrightarrow{d_{n-1}} A_n),\ n \in \mathbb{Z}.$$

(ii) *A morphism of complexes $A'_\bullet \xrightarrow{h_\bullet} A_\bullet$ is a commutative diagram*

$$\begin{tikzcd}
\cdots \arrow[r] & A'_{n-1} \arrow[r, "d'_{n-1}"] \arrow[d, "h_{n-1}"] & A'_n \arrow[r, "d'_n"] \arrow[d, "h_n"] & A'_{n+1} \arrow[r] \arrow[d, "h_{n+1}"] & \cdots \\
\cdots \arrow[r] & A_{n-1} \arrow[r, "d_{n-1}"] & A_n \arrow[r, "d_n"] & A_{n+1} \arrow[r] & \cdots
\end{tikzcd}$$

(iii) *Two morphisms of complexes*

$$A'_\bullet \underset{k_\bullet}{\overset{h_\bullet}{\rightrightarrows}} A_\bullet$$

are called "homotopy equivalent" if there exists a sequence of morphisms $\quad (A'_n \xrightarrow{b_n} A_{n-1})_{n\in\mathbb{Z}}$ *such that, for any $n \in \mathbb{Z}$,*

$$h_n - k_n = d_{n-1} \circ b_n + b_{n+1} \circ d'_n\,.$$

Derived categories of abelian categories

Lemma. –
Let $\mathcal{A}$ be an abelian category.
Then any morphism of complexes in $\mathcal{A}$

$$h_\bullet : A'_\bullet \longrightarrow A_\bullet$$

induces a sequence of morphisms

$$H^n(A'_\bullet) \longrightarrow H^n(A_\bullet)\,, \quad n \in \mathbb{Z}\,.$$

Furthermore, they only depend on the homotopy class of $h_\bullet$.

Definition. –
Let $\mathcal{A}$ be an abelian category.
Let $K(\mathcal{A})$ be the category of complexes in $\mathcal{A}$
and of homotopy classes of morphisms of complexes.

(i) *A morphism of $K(\mathcal{A})$*

$$A'_\bullet \longrightarrow A_\bullet$$

is called a "quasi-isomorphism" if it induces isomorphisms of $\mathcal{A}$

$$H^n(A'_\bullet) \xrightarrow{\sim} H^n(A_\bullet) \quad \text{for any } n \in \mathbb{Z}\,.$$

(ii) *The "derived category" of $\mathcal{A}$*

$$D(\mathcal{A})$$

is the category deduced from $K(\mathcal{A})$
by formally inverting quasi-isomorphisms.
In particular, $D(\mathcal{A})$ has the same objects as $K(\mathcal{A})$.

Structures of derived categories

Proposition. –
Let $\mathcal{A}$ be an abelian category.

(i) *The associated derived category $D(\mathcal{A})$ is additive.*

(ii) *It is endowed with additive functors*

$$[m] : D(\mathcal{A}) \longrightarrow D(\mathcal{A}) \,, \quad m \in \mathbb{Z} \,,$$

induced by

$$A_\bullet = (A_n)_{n\in\mathbb{Z}} \longmapsto A_\bullet[m] = (A_{n+m})_{n\in\mathbb{Z}} \,.$$

(iii) *It contains as additive subcategories*

$$D^+(\mathcal{A}) \,,\ D^-(\mathcal{A}) \,,\ D^b(\mathcal{A})$$

the full subcategories on complexes

$$A_\bullet$$

such that

$$H^n(A_\bullet) = 0 \quad \text{if} \quad \begin{cases} n & \text{is outside some} \quad [m, +\infty[, \\ n & \text{is outside some} \quad]-\infty, m], \\ n & \text{is outside some} \quad [m_1, m_2]. \end{cases}$$

(iv) *In $D(\mathcal{A})$, there is a well-defined notion of "distinguished triangles"*

$$A_\bullet \longrightarrow B_\bullet \longrightarrow C_\bullet \longrightarrow A_\bullet[1],$$

which induces long exact sequences $\cdots \to H^n(A_\bullet) \to H^n(B_\bullet) \to H^n(C_\bullet) \to H^{n+1}(A_\bullet) \to \cdots$

Furthermore, any morphism $A_\bullet \to B_\bullet$ can be completed in a "distinguished triangle".

Cohomology of push-forward functors

Theorem. –
Consider a geometric category of "ring" type $\mathbb{T}_r$

$$\begin{array}{rcl} \mathcal{X} & \longrightarrow & \mathrm{Top}_{\mathbb{T}_r}\,, \\ X & \longmapsto & (\mathcal{E}_X, \mathcal{R}_X)\,, \\ (X \xrightarrow{f} Y) & \longmapsto & (\mathcal{E}_X \xrightarrow{(f^*, f_*)} \mathcal{E}_Y\,,\, f^*\mathcal{R}_Y \to \mathcal{R}_X)\,. \end{array}$$

Then:

(i) *For any morphism $X \xrightarrow{f} Y$ of $\mathcal{X}$, the linear functor*
$$f_* : \mathcal{M}od_{\mathcal{R}_X} \longrightarrow \mathcal{M}od_{\mathcal{R}_Y}$$
respects limits and induces an additive "derived functor"
$$Rf_* : D^+(\mathcal{M}od_{\mathcal{R}_X}) \longrightarrow D^+(\mathcal{M}od_{\mathcal{R}_Y})$$
which respects distinguished triangles.

(ii) *If f_* has "cohomological dimension $\leq d$"*
in the sense that
Rf_* transforms complexes vanishing outside $[m_1, m_2]$
into complexes vanishing outside $[m_1, m_2 + d]$,
then it extends to a full derived functor
$$Rf_* : D(\mathcal{M}od_{\mathcal{R}_X}) \longrightarrow D(\mathcal{M}od_{\mathcal{R}_Y}).$$

(iii) *This construction is functorial in the sense that*
$$R(g \circ f)_* \cong Rg_* \circ Rf_* \quad \text{for any} \quad X \xrightarrow{f} Y \xrightarrow{g} Z \text{ in } \mathcal{X}.$$

Cohomology of pull-back functors

Theorem. –
Consider a geometric category of "ring" type $\mathbb{T}_r$

$$\begin{array}{ccc} \mathcal{X} & \longrightarrow & \mathrm{Top}_{\mathbb{T}_r}\,, \\ (X \xrightarrow{f} Y) & \longmapsto & (\mathcal{E}_X \xrightarrow{(f^*,f_*)} \mathcal{E}_Y\,,\, f^*\mathcal{R}_Y \to \mathcal{R}_X)\,. \end{array}$$

Then:

(i) *For any morphism $X \xrightarrow{f} Y$ of $\mathcal{X}$, the linear functor*

$$\mathcal{R}_X \otimes_{f^*\mathcal{R}_Y} f^*(\bullet) : \mathcal{M}od_{\mathcal{R}_Y} \longrightarrow \mathcal{M}od_{\mathcal{R}_X}$$

respects colimits and induces an additive "derived functor"

$$Lf^* = \mathcal{R}_X \overset{L}{\otimes}_{f^*\mathcal{R}_Y} f^*(\bullet) : D^-(\mathcal{M}od_{\mathcal{R}_Y}) \longrightarrow D^-(\mathcal{M}od_{\mathcal{R}_X})$$

which respects distinguished triangles.

(ii) *If $\mathcal{R}_X \otimes_{f^*\mathcal{R}_Y} (\bullet)$ has "cohomological dimension $\leq d$"*
in the sense that

$\mathcal{R}_X \overset{L}{\otimes}_{f^*\mathcal{R}_Y} (\bullet)$ *transforms complexes vanishing outside $[m_1, m_2]$*
into complexes vanishing outside $[m_1 - d, m_2]$,

then Lf^* extends to a full derived functor

$$Lf^* : D(\mathcal{M}od_{\mathcal{R}_Y}) \longrightarrow D(\mathcal{M}od_{\mathcal{R}_X}).$$

(iii) *The functor Lf^* is left-adjoint to Rf_**
and its construction is functorial in the sense that $L(g \circ f)^* \cong Lf^* \circ Lg^*$.

Cohomology of tensor products in toposes

Theorem. –
Consider a topos $\mathcal{E}$ endowed with an inner commutative ring $\mathcal{O}$.
Then:

(i) *The bilinear functor of tensor product over $\mathcal{O}$*

$$(\bullet \otimes_{\mathcal{O}} \bullet) : \mathcal{M}od_{\mathcal{O}} \times \mathcal{M}od_{\mathcal{O}} \longrightarrow \mathcal{M}od_{\mathcal{O}}$$

respects colimits and induces an additive "derived functor"

$$(\bullet \overset{L}{\otimes}_{\mathcal{O}} \bullet) : D^{-}(\mathcal{M}od_{\mathcal{O}}) \times D(\mathcal{M}od_{\mathcal{O}}) \longrightarrow D(\mathcal{M}od_{\mathcal{O}})$$

which respects distinguished triangles.

(ii) *If $(\bullet \otimes_{\mathcal{O}} \bullet)$ has "cohomological dimension $\leq d$"*
in the sense that, for any object $\mathcal{M}$ of $\mathcal{M}od_{\mathcal{O}}$,

$\bullet \overset{L}{\otimes}_{\mathcal{O}} \mathcal{M}$ *transforms complexes vanishing outside $[m_1, m_2]$*
into complexes vanishing outside $[m_1 - d, m_2]$,

then $\bullet \overset{L}{\otimes}_{\mathcal{O}} \bullet$ *extends to a full derived functor*

$$(\bullet \overset{L}{\otimes}_{\mathcal{O}} \bullet) : D(\mathcal{M}od_{\mathcal{O}}) \times D(\mathcal{M}od_{\mathcal{O}}) \longrightarrow D(\mathcal{M}od_{\mathcal{O}}).$$

(iii) *The derived functor*

$$\bullet \overset{L}{\otimes}_{\mathcal{O}} \bullet$$

is associative in the sense that

$$(\bullet \overset{L}{\otimes}_{\mathcal{O}} \bullet) \overset{L}{\otimes}_{\mathcal{O}} \bullet \cong \bullet \overset{L}{\otimes}_{\mathcal{O}} (\bullet \overset{L}{\otimes}_{\mathcal{O}} \bullet).$$

Cohomology of linear functionals in toposes

Theorem. –
Consider a topos $\mathcal{E}$ endowed with an inner commutative ring $\mathcal{O}$.
Then:

(i) *The bilinear functor of local linear functionals*

$$\mathcal{H}om_{\mathcal{O}} : \mathcal{M}od_{\mathcal{O}}^{\mathrm{op}} \times \mathcal{M}od_{\mathcal{O}} \longrightarrow \mathcal{M}od_{\mathcal{O}}$$

respects limits in the second variable and induces an additive "derived functor"

$$R\,\mathcal{H}om_{\mathcal{O}} : D(\mathcal{M}od_{\mathcal{O}})^{\mathrm{op}} \times D^{+}(\mathcal{M}od_{\mathcal{O}}) \longrightarrow D(\mathcal{M}od_{\mathcal{O}})$$

which respects distinguished triangles.

(ii) *The derived functors*

$$(\bullet \overset{L}{\otimes} \bullet) \quad \text{and} \quad R\,\mathcal{H}om_{\mathcal{O}}(\bullet, \bullet)$$

are adjoint in the sense that for any objects

$\mathcal{M}_\bullet$ *of* $D^{-}(\mathcal{M}od_{\mathcal{O}})$ *(or $D(\mathcal{M}od_{\mathcal{O}})$ if $\overset{L}{\otimes}_{\mathcal{O}}$ has bounded cohomological dimensions),*
$\mathcal{M}'_\bullet$ *of* $D(\mathcal{M}od_{\mathcal{O}})$,
$\mathcal{M}''_\bullet$ *of* $D^{+}(\mathcal{M}od_{\mathcal{O}})$,

$$\mathrm{Hom}(\mathcal{M}_\bullet \overset{L}{\otimes}_{\mathcal{O}} \mathcal{M}'_\bullet, \mathcal{M}''_\bullet) \quad \text{identifies with} \quad \mathrm{Hom}(\mathcal{M}_\bullet, R\,\mathcal{H}om_{\mathcal{O}}(\mathcal{M}'_\bullet, \mathcal{M}''_\bullet)).$$

The general geometric context for local and global cohomology:

Consider a category $\mathcal{X}$ endowed with a functor

$$\begin{array}{ccc} \mathcal{X} & \longrightarrow & \mathrm{Top}_{\mathbb{T}_{cr}} \, , \\ X & \longmapsto & (\mathcal{E}_X, \mathcal{O}_X) \, , \\ (X \xrightarrow{f} Y) & \longmapsto & (\mathcal{E}_X \xrightarrow{(f^*, f_*)} \mathcal{E}_Y \, , \, f^* \mathcal{O}_Y \to \mathcal{O}_X) \end{array}$$

where

$\mathrm{Top}_{\mathbb{T}_{cr}}$ = category of toposes $\mathcal{E}$

endowed with an "inner commutative ring" $\mathcal{O}$

i.e. a model $\mathcal{O}$ in the topos $\mathcal{E}$

of the theory $\mathbb{T}_{cr}$ of commutative rings.

Local operations and associated local cohomology

Consider

$$\begin{cases} \bullet \quad \text{a } \underline{\text{topos}}\ \mathcal{E}, \\ \bullet \quad \text{an "}\underline{\text{inner commutative ring}}\text{" } \mathcal{O} \text{ of } \mathcal{E}, \\ \bullet \quad \text{the } \underline{\text{associated abelian category}} \text{ of } \underline{\text{inner modules}} \text{ over } \mathcal{O} \text{ in } \mathcal{E} \\ \qquad\qquad \mathcal{M}od_{\mathcal{O}}\,. \end{cases}$$

The category $\mathcal{M}od_{\mathcal{O}}$ is endowed with the <u>bilinear functors</u>

$$\begin{cases} \bullet \otimes_{\mathcal{O}} \bullet & : \quad \mathcal{M}od_{\mathcal{O}} \times \mathcal{M}od_{\mathcal{O}} \longrightarrow \mathcal{M}od_{\mathcal{O}}\,, \\ \mathcal{H}om_{\mathcal{O}}(\bullet, \bullet) & : \quad \mathcal{M}od_{\mathcal{O}}^{\mathrm{op}} \times \mathcal{M}od_{\mathcal{O}} \longrightarrow \mathcal{M}od_{\mathcal{O}}\,. \end{cases}$$

They induce <u>derived functors</u>

$$\bullet \overset{L}{\otimes}_{\mathcal{O}} \bullet : D^{-}(\mathcal{M}od_{\mathcal{O}}) \times D(\mathcal{M}od_{\mathcal{O}}) \longrightarrow D(\mathcal{M}od_{\mathcal{O}})$$

[or even

$$D(\mathcal{M}od_{\mathcal{O}}) \times D(\mathcal{M}od_{\mathcal{O}}) \longrightarrow D(\mathcal{M}od_{\mathcal{O}})],$$

$$R\,\mathcal{H}om_{\mathcal{O}}(\bullet, \bullet) : D(\mathcal{M}od_{\mathcal{O}})^{\mathrm{op}} \times D^{+}(\mathcal{M}od_{\mathcal{O}}) \longrightarrow D(\mathcal{M}od_{\mathcal{O}}).$$

Ordinary global operations and associated global cohomology

Consider

$$\begin{cases} \bullet \quad \text{a \underline{topos morphism} } \mathcal{E}' \xrightarrow{(f^*, f_*)} \mathcal{E}, \\ \bullet \quad \text{“\underline{inner commutative rings}” } \mathcal{O}' \text{ in } \mathcal{E}' \text{ and } \mathcal{O} \text{ in } \mathcal{E} \\ \quad \text{related by a \underline{ring morphism}} \\ \quad f^*\mathcal{O} \longrightarrow \mathcal{O}' \quad \text{or, equivalently,} \quad \mathcal{O} \longrightarrow f_*\mathcal{O}' . \end{cases}$$

Then, the abelian categories $\mathcal{M}od_{\mathcal{O}}$ and $\mathcal{M}od_{\mathcal{O}'}$ are related by adjoint linear functors

$$f^* = \mathcal{O}' \otimes_{f^*\mathcal{O}} f^*(\bullet) : \mathcal{M}od_{\mathcal{O}} \longrightarrow \mathcal{M}od_{\mathcal{O}'}$$

and

$$f_* : \mathcal{M}od_{\mathcal{O}'} \longrightarrow \mathcal{M}od_{\mathcal{O}}.$$

They induce derived functors

$$Lf^* : D^-(\mathcal{M}od_{\mathcal{O}}) \longrightarrow D^-(\mathcal{M}od_{\mathcal{O}'})$$

[or even

$$D(\mathcal{M}od_{\mathcal{O}}) \longrightarrow D(\mathcal{M}od_{\mathcal{O}'})],$$

$$Rf_* : D^+(\mathcal{M}od_{\mathcal{O}'}) \longrightarrow D^+(\mathcal{M}od_{\mathcal{O}})$$

[or even

$$D(\mathcal{M}od_{\mathcal{O}'}) \longrightarrow D(\mathcal{M}od_{\mathcal{O}})$$

if f_* has “finite cohomological dimension”].

An important distinction of Grothendieck: "discrete" or "continuous" coefficients

"Discrete coefficients":
This is the situation of a functor

$$\begin{array}{ccc} \mathcal{X} & \longrightarrow & \mathrm{Top}_{\mathbb{T}_{cr}}\,, \\ (X \xrightarrow{f} Y) & \longmapsto & (\mathcal{E}_X \xrightarrow{(f^*,f_*)} \mathcal{E}_Y\,,\, f^*\mathcal{O}_Y \to \mathcal{O}_X) \end{array}$$

such that, for any arrow $f : X \to Y$ *in* $\mathcal{X}$*,*

$$f^*\mathcal{O}_Y \longrightarrow \mathcal{O}_X \qquad \textit{is an isomorphism.}$$

Remark:
If $\mathcal{X}$ has a "terminal object" X_0,

i.e. any object X of $\mathcal{X}$ has a unique morphism $X \xrightarrow{p_X} X_0$,
this is the situation of a functor to the category of toposes

$$\begin{array}{ccc} \mathcal{X} & \longrightarrow & \mathrm{Top} \\ (X \xrightarrow{f} Y) & \longmapsto & (\mathcal{E}_X \xrightarrow{(f^*,f_*)} \mathcal{E}_Y), \end{array}$$

and the choice of an inner commutative ring $\mathcal{O}_{X_0}$ in $\mathcal{E}_{X_0}$,
inducing $\qquad \mathcal{O}_X = p_X^*\mathcal{O}_{X_0}, \quad \forall\, X = \text{object of } \mathcal{X}.$

Examples where $\mathcal{X} \to \mathrm{Top}$ is fully faithful:

$$\{\text{sober topological spaces}\} \xrightarrow{X \mapsto \mathcal{E}_X} \mathrm{Top},$$

$$\{\text{finitely presentable algebraic varieties over } \mathbb{Q}\} \xrightarrow{X \mapsto \mathcal{E}_X^{\text{ét}}} \mathrm{Top}$$

(work of Carlson, Haine and Wolf).

"Continuous coefficients"

This is the situation of a functor

$$\begin{array}{ccc} \mathcal{X} & \longrightarrow & \mathrm{Top}_{\mathbb{T}_{cr}}\,, \\ (X \xrightarrow{f} Y) & \longmapsto & (\mathcal{E}_X \xrightarrow{(f^*, f_*)} \mathcal{E}_Y\,,\, f^*\mathcal{O}_Y \to \mathcal{O}_X) \end{array}$$

where, for any object X of $\mathcal{X}$,
$\mathcal{O}_X$ can be seen as part of the structure of X as an object of $\mathcal{X}$.
Typically, this is the situation where the functor

$$\mathcal{X} \longrightarrow \mathrm{Top}_{\mathbb{T}_{cr}}$$

is faithful or even factorizes as a fully faithful functor

$$\mathcal{X} \longrightarrow \mathrm{Top}_{\mathbb{T}_{\ell cr}}$$

to the category of toposes $\mathcal{E}$ endowed with a model $\mathcal{O}$
of the theory $\mathbb{T}_{\ell cr}$ of "local commutative rings"
(or the theory of "local commutative algebras" over $\mathbb{R}$ or $\mathbb{C} \cdots$).

Examples:

- Diff = category of differential manifolds
  $X \longmapsto (\mathcal{E}_X, \mathcal{O}_X =$ sheaf of ∞-differentiable functions).
- An = category of analytic varieties
  $X \longmapsto (\mathcal{E}_X, \mathcal{O}_X =$ sheaf of holomorphic functions).
- Sch = category of schemes
  $X \longmapsto (\mathcal{E}_X =$ Zariski topos, $\mathcal{O}_X =$ structure sheaf).

Squarable morphisms

Definition. –

A morphism $X \to Y$ in a cateogry $\mathcal{X}$
is called "squarable"
if any morphism $Y' \to Y$ defines a cartesian square in $\mathcal{X}$:

$$\begin{tikzcd} X' = X \times_Y Y' \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & X \arrow[d] \\ Y' \arrow[r] & Y \end{tikzcd}$$

Remark:
In such a situation, $X \times_Y Y' \to Y'$ is said to be induced from $X \to Y$ by the "base change" morphism $Y' \to Y$.

Definition. –
A class of morphisms in a category $\mathcal{X}$ is called "geometric" if:

(1) *This class contains isomorphisms.*

(2) *It is stable under composition.*

(3) *Elements of this class are squarable,*
and the morphisms they induce by base change still belong to this class.

Cartesian squares and commutative squares

Consider a geometric category

$$\begin{array}{rcl} \mathcal{X} & \longrightarrow & \mathrm{Top}_{\mathbb{T}_{cr}}\,, \\ X & \longmapsto & (\mathcal{E}_X, \mathcal{O}_X)\,, \\ (X \xrightarrow{f} Y) & \longmapsto & (\mathcal{E}_X \xrightarrow{(f^*, f_*)} \mathcal{E}_Y\,,\, f^*\mathcal{O}_Y \to \mathcal{O}_X). \end{array}$$

Then, any cartesian square in $\mathcal{X}$

$$\begin{tikzcd} X' = X \times_Y Y' \arrow[r, "x"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ Y' \arrow[r, "y"] & Y \end{tikzcd}$$

or more generally any commutative square in $\mathcal{X}$

$$\begin{tikzcd} X' \arrow[r] \arrow[d] & X \arrow[d] \\ Y' \arrow[r] & Y \end{tikzcd}$$

defines a commutative square in $\mathrm{Top}_{\mathbb{T}_{cr}}$:

$$\begin{tikzcd} (\mathcal{E}_{X'}, \mathcal{O}_{X'}) \arrow[r] \arrow[d] & (\mathcal{E}_X, \mathcal{O}_X) \arrow[d] \\ (\mathcal{E}_{Y'}, \mathcal{O}_{Y'}) \arrow[r] & (\mathcal{E}_Y, \mathcal{O}_Y) \end{tikzcd}$$

Cartesian squares and associated base change transforms on sheaves

Consider a cartesian or more generally commutative square in a geometric category $\mathcal{X} \to \mathrm{Top}_{\mathbb{T}_{cr}}$

$$\begin{tikzcd} X' \arrow[r, "x"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ Y' \arrow[r, "y"] & Y \end{tikzcd}$$

and the associated commutative square in $\mathrm{Top}_{\mathbb{T}_{cr}}$:

$$\begin{tikzcd} (\mathcal{E}_{X'}, \mathcal{O}_{X'}) \arrow[r] \arrow[d] & (\mathcal{E}_X, \mathcal{O}_X) \arrow[d] \\ (\mathcal{E}_{Y'}, \mathcal{O}_{Y'}) \arrow[r] & (\mathcal{E}_Y, \mathcal{O}_Y) \end{tikzcd}$$

Lemma. – *The commutative identification* $\quad f_* \circ x_* \xrightarrow{\sim} y_* \circ f'_*$
and the adjunctions (y^*, y_*) *and* (x^*, x_*)*, yield natural transformations of functors:*

(i) *From*

$$\mathcal{M}od_{\mathcal{O}_X} \xrightarrow{f_*} \mathcal{M}od_{\mathcal{O}_Y} \xrightarrow{y^*} \mathcal{M}od_{\mathcal{O}_{Y'}}$$

to

$$\mathcal{M}od_{\mathcal{O}_X} \xrightarrow{x^*} \mathcal{M}od_{\mathcal{O}_{X'}} \xrightarrow{f'_*} \mathcal{M}od_{\mathcal{O}_{Y'}}.$$

(ii) *From*

$$D^-(\mathcal{M}od_{\mathcal{O}_X}) \xrightarrow{Rf_*} D^-(\mathcal{M}od_{\mathcal{O}_Y}) \xrightarrow{Ly^*} D^-(\mathcal{M}od_{\mathcal{O}_{Y'}})$$

to

$$D^-(\mathcal{M}od_{\mathcal{O}_X}) \xrightarrow{Lx^*} D^-(\mathcal{M}od_{\mathcal{O}_{X'}}) \xrightarrow{Rf'_*} D^-(\mathcal{M}od_{\mathcal{O}_{Y'}})$$

if f and f' have bounded cohomological dimensions.

Proper morphisms, base change and extraordinary pull-back

Discovery (Grothendieck). – *For geometric categories coming from concrete practice*

$$\mathcal{X} \longrightarrow \mathrm{Top}_{\mathbb{T}_{cr}} ,$$

there is generally a geometric class of "proper morphisms" of $\mathcal{X}$ such that:

(i) *Proper morphisms have bounded cohomological dimension, and they define functors*

$$(X \xrightarrow{f} Y) \longmapsto (D(\mathcal{M}od_{\mathcal{O}_X}) \xrightarrow{Rf_*} D(\mathcal{M}od_{\mathcal{O}_Y})).$$

(ii) *For any cartesian square on a proper morphism $X \xrightarrow{f} Y$*

$$\begin{tikzcd} X' = X \times_Y Y' \arrow[r, "x"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ Y' \arrow[r, "y"] & Y \end{tikzcd}$$

the associated base change natural transform

$$Ly^* \circ Rf_* \longrightarrow Rf'_* \circ Lx^*$$

is an isomorphism.

(iii) *If $X \xrightarrow{f} Y$ is a proper morphism, the derived functor*

$$f_! = Rf_* : D^+(\mathcal{M}od_{\mathcal{O}_X}) \longrightarrow D^+(\mathcal{M}od_{\mathcal{O}_Y})$$

has a right adjoint

$$f^! : D^+(\mathcal{M}od_{\mathcal{O}_Y}) \longrightarrow D^+(\mathcal{M}od_{\mathcal{O}_X}).$$

Smooth morphisms for base change

Discovery (Grothendieck). –
For geometric categories coming from concrete practice

$$\mathcal{X} \longrightarrow \mathrm{Top}_{\mathbb{T}_{cr}}\,,$$

there is generally a geometric class of "smooth morphisms" of $\mathcal{X}$ such that:

For any morphism $X \xrightarrow{f} Y$ *in* $\mathcal{X}$,

any smooth morphism $Y' \xrightarrow{y} Y$

and the cartesian square they define

$$\begin{tikzcd} X' = X \times_Y Y' \arrow[r, "x"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ Y' \arrow[r, "y"] & Y \end{tikzcd}$$

the associated base change natural transform

$$Ly^* \circ Rf_* \longrightarrow Rf'_* \circ Lx^*$$

is an isomorphism.

Remark:
The base change stability properties which characterize proper morphisms and smooth morphisms are in some sense dual to each other.

Open morphisms

Discovery (Grothendieck). – *For geometric categories coming from concrete practice*

$$\mathcal{X} \longrightarrow \mathrm{Top}_{\mathbb{T}_{cr}}$$

(with "discrete coefficients", or even with "continuous coefficients" after some modifications of the categories $\mathcal{M}od_{\mathcal{O}_X}$ and $D(\mathcal{M}od_{\mathcal{O}_X})$), there is generally a geometric class of "open morphisms" of $\mathcal{X}$ such that:

(i) *For open morphisms* $U \xrightarrow{i} X$, *the induced functor*

$$i^! = i^* : \mathcal{M}od_{\mathcal{O}_X} \longrightarrow \mathcal{M}od_{\mathcal{O}_U}$$

has a left adjoint functor

$$i_! : \mathcal{M}od_{\mathcal{O}_U} \longrightarrow \mathcal{M}od_{\mathcal{O}_X}$$

whis is exact, defining a functor

$$i_! : D^+(\mathcal{M}od_{\mathcal{O}_U}) \longrightarrow D^+(\mathcal{M}od_{\mathcal{O}_X})$$

left adjoint to $i^! = i^* : D^+(\mathcal{M}od_{\mathcal{O}_X}) \to D^+(\mathcal{M}od_{\mathcal{O}_U})$.

(ii) *For any cartesian square on an open morphism* $U \to X$

$$\begin{tikzcd} U' = U \times_X X' \arrow[r, "u"] \arrow[d, "i'"'] & U \arrow[d, "i"] \\ X' \arrow[r, "x"] & X \end{tikzcd}$$

the associated natural transform

$$i'_! \circ Lu^* \longrightarrow Lx^* \circ i_!$$

is an isomorphism.

Extraordinary operations for factorizable morphisms

Lemma. – *Consider a geometric category $\mathcal{X}$ with geometric classes of "proper morphisms" and "open morphisms". Then the class of "compactifiable morphisms", i.e. of composites*

$$U \xrightarrow{\text{open}} X \xrightarrow{\text{proper}} Y\,,$$

is geometric.

Discovery (Grothendieck). – *For geometric categories coming from concrete practice*

$$\mathcal{X} \longrightarrow \mathrm{Top}_{\mathbb{T}_{cr}}$$

with geometric classes of "proper morphisms" and "open morphisms",
the associated geometric class of "compactifiable morphisms" verifies:

(i) *For any morphism of this class factorized as* $\quad f : U \xrightarrow[\text{open}]{i} X \xrightarrow[\text{proper}]{\bar{f}} Y\,,$

the functor $f_! = \bar{f}_! \circ i_! : D^+(\mathcal{M}od_{\mathcal{O}_U}) \to D^+(\mathcal{M}od_{\mathcal{O}_Y})$
and its right adjoint $\quad f^! = i^! \circ \bar{f}^! \quad$ do not depend on the factorization of f.

(ii) *For any cartesian square on a compactifiable morphism $f : U \to Y$*

$$\begin{tikzcd} U' = U \times_Y Y' \arrow[r, "u"] \arrow[d, "f'"'] & U \arrow[d, "f"] \\ Y' \arrow[r, "y"] & Y \end{tikzcd}$$

there is a natural isomorphism $\quad f'_! \circ Lu^* \xrightarrow{\sim} Ly^* \circ f_!.$

Global duality according to Grothendieck

Discovery (Grothendieck). – *For geometric categories coming from concrete practice*

$$\mathcal{X} \longrightarrow \mathrm{Top}_{\mathbb{T}_{cr}}$$

with geometric classes of "proper morphisms", "open morphisms" and an associated class of "compactifiable morphisms" $U \xrightarrow{f} Y$ *defining a pair of adjoint "extraordinary functors"*

$$(D^+(\mathcal{M}od_{\mathcal{O}_U}) \xrightarrow{f_!} D^+(\mathcal{M}od_{\mathcal{O}_Y}), D^+(\mathcal{M}od_{\mathcal{O}_Y}) \xrightarrow{f^!} D^+(\mathcal{M}od_{\mathcal{O}_X})),$$

this construction verifies the following properties:

(i) *It is functorial in the sense that for any pair of compactifiable morphisms*

$$X \xrightarrow{f} Y \xrightarrow{g} Z,$$

there are canonical isomorphisms

$$(g \circ f)_! \xrightarrow{\sim} g_! \circ f_!,$$
$$(g \circ f)^! \xrightarrow{\sim} f^! \circ g^!.$$

(ii) *For any compactifiable morphism* $X \xrightarrow{f} Y$*, objects* $\mathcal{L}$ *of* $D^+(\mathcal{M}od_{\mathcal{O}_Y})$ *define functorially commutative squares of "global duality" between the functors* $f_!$ *and* Rf_**:*

$$\begin{tikzcd}
D^+(\mathcal{M}od_{\mathcal{O}_X})^{\mathrm{op}} \arrow[r, "R\mathcal{H}om_{\mathcal{O}_X}(\bullet, f^!\mathcal{L})"] \arrow[d, "f_!"] & D(\mathcal{M}od_{\mathcal{O}_X}) \arrow[d, "Rf_*"] \\
D^+(\mathcal{M}od_{\mathcal{O}_Y})^{\mathrm{op}} \arrow[r, "R\mathcal{H}om_{\mathcal{O}_Y}(\bullet, \mathcal{L})"] & D(\mathcal{M}od_{\mathcal{O}_Y})
\end{tikzcd}$$

Local duality according to Grothendieck

Proposition. –
Consider a topos $\mathcal{E}$ endowed with an inner commutative ring $\mathcal{O}$.
Then objects $\mathcal{L}$ of $D^+(\mathcal{M}od_{\mathcal{O}})$ functorially define natural transformations of functors on $D(\mathcal{M}od_{\mathcal{O}})$

$$\mathrm{Id}_{D(\mathcal{M}od_{\mathcal{O}})} \longrightarrow R\mathcal{H}om_{\mathcal{O}}(R\mathcal{H}om_{\mathcal{O}}(\bullet, \mathcal{L}), \mathcal{L}).$$

Discovery of the "biduality" phenomenon (Grothendieck). –
For geometric categories coming from concrete practice, $\mathcal{X} \longrightarrow \mathrm{Top}_{\mathbb{T}_{cr}}$
with geometric classes of "proper morphisms", "open morphisms", and "compactifiable morphisms" defining "extraordinary" adjoint pairs $(f_!, f^!)$,
then it is generally possible to associate to objects X of $\mathcal{X}$

- *a distinguished full subcategory $D^b(\mathcal{M}od_{\mathcal{O}_X})_c$ of $D^b(\mathcal{M}od_{\mathcal{O}_X})$ consisting of "constructible objects",*
- *a non-trivial class of so-called "dualizing objects" in $D^+(\mathcal{M}od_{\mathcal{O}})$,*

such that:

(1) *The six operations $\overset{L}{\otimes}$, $R\mathcal{H}om(\bullet, \bullet)$, $f^*, f_*, f_!, f^!$ transform "constructible objects" into "constructible objects".*

(2) *The operations $f^!$ transform "dualizing objects" into "dualizing objects".*

(3) *If $\mathcal{M}$ is a "constructible object" and $\mathcal{L}$ is a "dualizing object", the natural morphism*
$$\mathcal{M} \longrightarrow R\mathcal{H}om_{\mathcal{O}}(R\mathcal{H}om_{\mathcal{O}}(\mathcal{M}, \mathcal{L}), \mathcal{L})$$
is an isomorphism.

An illustation of Grothendieck's duality

Observation. –
For any geometric category $\mathcal{X} \longrightarrow \mathrm{Top}_{\mathbb{T}_{cr}}$ *with a well-defined "six operations formalism", any "compactifiable morphism"* $X \xrightarrow{f} Y$ *of* $\mathcal{X}$ *has an associated "extraordinary" adjoint pair*

$$(D^+(\mathcal{M}od_{\mathcal{O}_X}) \xrightarrow{f_!} D^+(\mathcal{M}od_{\mathcal{O}_Y}),\ D^+(\mathcal{M}od_{\mathcal{O}_Y}) \xrightarrow{f^!} D^+(\mathcal{M}od_{\mathcal{O}_X})),$$

which implies that there is a natural transform of functors

$$\mathrm{Tr} : f_! f^! \longrightarrow \mathrm{Id} \qquad \textit{called the "trace transform".}$$

Example:
Consider the geometric category $\mathcal{X} = \mathrm{Diff}$ of differential manifolds.

If a morphism $X \xrightarrow{f} Y$ is submersive
(in the sense that its differential at any point is surjective),
it is squarable and defines an "extraordinary" adjoint pair

$$(f_!, f^!).$$

If X is a d-dim. oriented manifold and $f = p : X \to \{\bullet\} = Y$ is the projection to the one-point manifold, one verifies that

- the image by $f^!$ of the sheaf $\mathbb{R}$ is the complex concentrated in degree d which consists in the sheaf Ω_X of differential forms of degree d on X,
- the trace morphism

$$f_!\,\Omega_X \longrightarrow \mathbb{R}$$

is just integration of differential forms of degree d with compact support!

A general question of Grothendieck

In "Récoltes et Semailles",

Grothendieck raised the following general problem:

Problem:

Identify general topossic conditions

for the existence of a "six operations formalism".

Proposed precisions:

(i) For us the general setting of this problem is a category $\mathcal{X}$ endowed with a functor

$$\mathcal{X} \longrightarrow \mathrm{Top}_{\mathbb{T}_{cr}}$$

which can be requested to be faithful.

(ii) A "six operations formalism" would consist in

- geometric classes of "proper" morphisms, "smooth" morphisms, "open" morphisms, "compactifiable" morphisms,
- a functorial construction on "compactifiable" morphisms

$$f \longmapsto (f_!, f^!)$$

- notions of "constructible" objects and "dualizing" objects, which would verify all properties discovered by Grothendieck.

A triple question on morphisms of toposes

The general problem raised by Grothendieck begins with the following questions:

Problem:
Consider a morphism of toposes endowed with inner commutative rings

$$(\mathcal{E}' \xrightarrow{(f^*, f_*)} \mathcal{E}\,,\ f^*\mathcal{O} \longrightarrow \mathcal{O}').$$

(i) *Are there general geometric conditions for the induced functor*

$$f_* : \mathcal{M}od_{\mathcal{O}'} \longrightarrow \mathcal{M}od_{\mathcal{O}}$$

to have bounded cohomological dimension?

(ii) *Are there general geometric conditions for the induced functor*

$$Rf_* : D^+(\mathcal{M}od_{\mathcal{O}'}) \longrightarrow D^+(\mathcal{M}od_{\mathcal{O}})$$

to have a right adjoint

$$f^! : D^+(\mathcal{M}od_{\mathcal{O}}) \longrightarrow D^+(\mathcal{M}od_{\mathcal{O}'}) \quad ?$$

(iii) *Are there general geometric conditions for the induced functors*

$$\begin{aligned} Rf_* : D^+(\mathcal{M}od_{\mathcal{O}'}) &\longrightarrow D^+(\mathcal{M}od_{\mathcal{O}})\,, \\ Lf^* : D^-(\mathcal{M}od_{\mathcal{O}}) &\longrightarrow D^-(\mathcal{M}od_{\mathcal{O}'}) \end{aligned}$$

to transform objects which can be presented by finite data
into objects which can be presented by finite data?

Remark:
Indeed, "constructible" objects are often in practice
"objects which can be presented by finite data", at least locally.

A general question for base change compatibility

In any "six operations formalism" on a geometric category

$$\mathcal{X} \longrightarrow \mathrm{Top}_{\mathbb{T}_{cr}}$$

there should be base change compatibility

for all "proper" morphisms

and by all "smooth" morphisms.

A difficulty is that, in general, the functor

$$\mathcal{X} \longrightarrow \mathrm{Top}_{\mathbb{T}_{cr}}$$

would not respect cartesian squares.

Nevertheless, one may ask:

Problem:

Consider a commutative square in the category $\mathrm{Top}_{\mathbb{T}_{cr}}$

$$\begin{tikzcd}
(\mathcal{E}', \mathcal{O}') \arrow[r, "f'"] \arrow[d, "e"'] & (\mathcal{B}', \mathcal{A}') \arrow[d, "b"] \\
(\mathcal{E}, \mathcal{O}) \arrow[r, "f"] & (\mathcal{B}, \mathcal{A})
\end{tikzcd}$$

Are there general geometric conditions for the natural base change transform

$$Lf^* \circ Rb_* \longrightarrow Re_* \circ Lf'^*$$

to be an isomorphism?