What is a point? What is a space?

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What is a real number?

We all know the classical constructions of $\mathbb{R}$ by Cauchy or Dedekind:

Definition. –
A real number is an equivalence class of sequences

$$n \in \mathbb{N} \longmapsto r_n \in \mathbb{Q}$$

which verify the "Cauchy condition"

$$\lim_{n_1, n_2 \to +\infty} |r_{n_1} - r_{n_2}| = 0\,,$$

for the equivalence relation

$$(r_n)_{n \in \mathbb{N}} \sim (r'_n)_{n \in \mathbb{N}}$$

if and only if

$$\lim_{n \to +\infty} |r_n - r'_n| = 0\,.$$

Definition. –
A real number is a subset

$$x \subseteq \mathbb{Q}$$

which is

$$\begin{cases} \bullet \quad \textit{non empty}, \\ \bullet \quad \textit{bounded from above}, \\ \bullet \quad \textit{complete from below in the sense that, for any } r \in x, \\ \qquad\qquad r' \leq r \quad \Rightarrow \quad r' \in x\,. \end{cases}$$

An alternate definition of real numbers by approximations

Here is an alternate definition of real numbers
which formalizes the idea that a real number
is something which can be approximated with arbitrary precision:

Definition. –
Let Q be an ordered set consisting in
a dense subset of $\mathbb{Q}$
completed with the symbols $-\infty$ and $+\infty$.
Then a real number x is a coherent family of answers
yes or no
to the question
"Does x belong to the interval $]a,b[$?"
for any pair of elements $a < b$ in Q,
if we call "coherent" a family of answers verifying:

$$\begin{cases} (F) & \text{If the answer for }]a,b[\text{ is "yes", it is "yes" for any }]a',b'[\,\supset\,]a,b[\,. \\ (C) & \text{If }]a,b[\,=\,]a',b'[\,\cap\,]a'',b''[\,, \\ & \text{the answer is "yes" for }]a,b[\text{ if it is "yes" for }]a',b'[\text{ and }]a'',b''[\,. \\ (C_1) & \text{The answer for }]-\infty,+\infty[\text{ is "yes".} \\ (J) & \text{If the answer is yes for }]a,b[\,, \text{ and }]a,b[\,= \bigvee_{i\in I}]a_i,b_i[\,, \\ & \text{then the answer is yes for at least one }]a_i,b_i[\,,\ i \in I. \end{cases}$$

Reformulation of the definition of real numbers

We observe that the one-element set $\{\bullet\}$ has two subsets

$$\emptyset \quad \text{and} \quad \{\bullet\}$$

which can represent the answers "no" and "yes".

Definition. –
If Q is a dense subset of $\mathbb{Q}$ completed with the symbols $-\infty$, $+\infty$,
a real number is a mapping

$$\begin{array}{rcl} \rho : \left\{ \begin{array}{c} \textit{intervals }]a,b[\\ \textit{with } a < b \textit{ in } Q \end{array} \right\} & \longrightarrow & \{\textit{subsets of } \{\bullet\}\}, \\ ]a,b[& \longmapsto & \rho(]a,b[), \end{array}$$

such that:

(F) ρ *transforms any inclusion* $]a,b[\subseteq]a',b'[$
into an embedding $\rho(]a,b[) \subseteq \rho(]a',b'[)$.

(C) ρ *transforms any intersection* $]a,b[=]a',b'[\cap]a'',b''[$
into an intersection $\rho(]a,b[) = \rho(]a',b'[) \cap \rho(]a'',b''[)$.

(C_1) ρ *transforms the maximal interval* $]-\infty,+\infty[$
into the maximal subset $\rho(]-\infty,+\infty[) = \{\bullet\}$,

(J) ρ *transforms any covering* $]a,b[= \bigvee\limits_{i \in I}]a_i,b_i[$
into a covering $\rho(]a,b[) = \bigcup\limits_{i \in I} \rho(]a_i,b_i[)$.

What is a covering?

Proposition. – *Let Q be a dense subset of $\mathbb{Q}$, completed with the symbols $-\infty, +\infty$.*
In order to define real numbers, the notion of covering by subintervals

$$]a,b[= \bigvee_{i\in I}]a_i,b_i[$$

has to be defined in the following way:

$$\begin{cases} \bullet \quad \text{A family }]a_i,b_i[\subseteq]a,b[\,,\ i\in I, \text{ is a covering} \\ \quad\ \text{if and only if, for any } a < a' < b' < b, \\ \quad\ \text{there exists in } I \text{ a finite family } i_1, i_2, \cdots, i_n \in I \\ \quad\ \text{such that} \\ \quad\ a_{i_1} < a'\,,\ b' < b_{i_n} \quad \text{and} \quad a_{i_{k+1}} < b_{i_k}\,,\ 1 \le k < n\,. \end{cases}$$

Formal properties of coverings:

(0) A family of subintervals $]a_i,b_i[\subseteq]a,b[\,,\ i\in I$, is a covering if and only if its associated "sieve" $\{]a',b'[\mid \exists\, i \in I,\ ]a',b'[\subseteq]a_i,b_i[\}$ is a covering.

(1) Any $]a,b[$ is a covering of itself.

(2) If $]a,b[= \bigvee_{i\in I}]a_i,b_i[$, then for any $]a',b'[\subset]a,b[$,

$$]a',b'[= \bigvee_{i\in I} \big(]a',b'[\cap]a_i,b_i[\big).$$

(3) If $]a,b[= \bigvee_{i\in I}]a_i,b_i[$ and $]a_i,b_i[= \bigvee_{j\in I_i}]a_{i,j},b_{i,j}[\,,\ \forall\, i\in I$,

then $\quad]a,b[= \bigvee_{i\in I}\bigvee_{j\in I_i}]a_{i,j},b_{i,j}[\,.$

Generalization to ordered sets

Definition. –
Consider a partially ordered set $(O, \leq)$ *such that*

$$\begin{cases} \bullet \quad O \text{ has a biggest element, denoted } 1, \\ \bullet \quad \text{any elements } a, b \in O \text{ has an inf } a \wedge b \text{ characterized by} \\ \qquad\qquad c \leq a \wedge b \quad \Leftrightarrow \quad c \leq a \quad \text{and} \quad c \leq b\,. \end{cases}$$

A Grothendieck topology J *on* $(O, \leq)$ *consists in a notion of*

$$\text{"covering"} \quad a = \bigvee_{i \in I} a_i$$

of elements $a \in O$ *by families of smaller elements* $(a_i \leq a)_{i \in I}$*,*
which verifies the following axioms:

$$\begin{cases} (0) & \text{A family } (a_i \leq a)_{i \in I} \text{ is a covering} \\ & \text{if and only if its associated "sieve" } \{a' \mid \exists\, i \in I\,,\ a' \leq a_i\} \\ & \text{is a covering.} \\ (1) & \text{For any } a \in O,\ a \leq a \text{ is a covering.} \\ (2) & \text{If } \ a = \bigvee\limits_{i \in I} a_i \quad \text{and} \quad a' \leq a, \\ & \text{then} \quad a' = \bigvee\limits_{i \in I} (a' \wedge a_i). \\ (3) & \text{If } \ a = \bigvee\limits_{i \in I} a_i \quad \text{and} \quad a_i = \bigvee\limits_{j \in I_i} a_{i,j}, \forall\, i \in I, \\ & \text{then} \quad a = \bigvee\limits_{\substack{i \in I \\ j \in I_i}} a_{i,j}. \end{cases}$$

Generalization of the notion of point

Definition. –

Let $(O, \leq, 1, \wedge)$ *be a* *partially ordered set* *with a* *biggest element* 1 *and an* *inf operator* $\wedge$*.*
Let J *be a* *Grothendieck topology*.
In this context, a *point* *in* *Grothendieck's sense* *is a* *mapping*

$$\begin{array}{rccl} \rho \;:\; & O & \longrightarrow & \{\textit{subsets of}\ \{\bullet\}\}\,, \\ & a & \longmapsto & \rho(a)\,, \end{array}$$

such that:

$$\begin{cases} (F) & \rho \textit{ transforms any relation} \quad a' \leq a \\ & \textit{into an inclusion} \quad \rho(a') \subseteq \rho(a). \\ (C) & \rho \textit{ transforms any inf relation} \quad c = a \wedge b \\ & \textit{into an intersection} \quad \rho(c) = \rho(a) \cap \rho(b). \\ (C_1) & \rho \textit{ transforms the biggest element } 1 \\ & \textit{into the biggest subset} \quad \rho(1) = \{\bullet\}. \\ (J) & \rho \textit{ transforms any } J\textit{-covering} \quad a = \bigvee_{i \in I} a_i \\ & \textit{into a covering} \quad \rho(a) = \bigcup_{i \in I} \rho(a_i). \end{cases}$$

The particular case of ordinary coverings of open subsets

Proposition. –

Let X be a topological space and O_X be the set of open subsets of X.
Let $O \subseteq O_X$ be a subset
which is "dense" (meaning any element of O_X is a union of elements of O)
and such that

$$\begin{cases} \bullet & \text{the total space } X \text{ is an element of } O, \\ \bullet & O \text{ is stable under finite intersections.} \end{cases}$$

Let J be the "ordinary topology" on O,
for which a family $(U_i \leq U)_{i \in I}$ is "covering"
if and only if

$$U = \bigcup_{i \in I} U_i \,.$$

Then:

(i) *Any element $x \in X$ defines a point of (O, J).*

(ii) *Conversely, if the topological space X is "sober",*
any point of (O, J) is defined by a unique element $x \in X$.

Remark:
Almost all topological spaces met in practice are sober.
In particular, Hausdorff spaces are sober.

Topologies defined by measures

Lemma. –

Let X be a sober topological space.
Let $O \subseteq O_X$ be a dense subset
which contains X and is stable under finite intersections.

Then a measure μ of open subsets of X
defines a topology J_μ on O
by deciding that a family $(U_i \subseteq U)_{i \in I}$
is "covering" if and only if it contains a countable subfamily

$$(U_{i_n} \subseteq U)_{n \in \mathbb{N}} \quad \text{with} \quad i_n \in I\,,\ \forall\, n\,,$$

such that

$$\mu\Big(\bigcup_{n \in \mathbb{N}} U_{i_n}\Big) = \mu(U)\,.$$

Proposition. –

In this situation, points of (O, J_μ)
correspond to elements $x \in X$
such that $\mu(\{x\}) \neq 0$.

Remark:

In many natural cases, (O, J_μ) has no point
even though it defines a space (as we shall see).

Reminder on categories

A category (= mathematical country) $\mathcal{C}$ consists in

- "objects" (= mathematical cities) $X, Y, \cdots$
- "morphisms" (= itineraries) between objects (= cities) $X \to Y$,
- a composition law of morphisms (= itineraries)

$$(X \xrightarrow{f} Y \xrightarrow{g} Z) \longmapsto (X \xrightarrow{g \circ f} Z)$$

which is associative and admits a unit element $\mathrm{id}_X : X \to X$ for any object (= city) X.

Definition. – *A category $\mathcal{C}$ is called "cartesian" if:*

(i) *It admits a "terminal object" $1_{\mathcal{C}}$ in the sense that any object X of $\mathcal{C}$ has a unique morphism $X \to 1_{\mathcal{C}}$.*

(ii) *Any pair of morphisms $X \xrightarrow{f} S$ and $S' \xrightarrow{s} S$ defines a "cartesian commutative square"*

$$\begin{tikzcd} X \times_S S' \arrow[r, "x"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ S' \arrow[r, "s"] & S \end{tikzcd}$$

characterized by the property that any "commutative square"

$$\begin{tikzcd} X' \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ S' \arrow[r, "s"] & S \end{tikzcd}$$

yields a unique factorization

$$X' \longrightarrow X \times_S S' .$$

Grothendieck topologies on cartesian categories

Definition. –
Consider a (small) category which is cartesian.
A Grothendieck topology J on $\mathcal{C}$ consists in a notion of covering of objects X of $\mathcal{C}$
by families of morphisms $\quad (X_i \xrightarrow{x_i} X)_{i \in I}$
which verifies the following axioms:

(0) *A family $(X_i \xrightarrow{x_i} X)_{i \in I}$ is a covering if and only if its associated "sieve"*
$$\{X' \xrightarrow{x} X \mid x \text{ admits a factorization } X' \longrightarrow X_i \xrightarrow{x_i} X \text{ for some } i \in I\}$$
is a covering.

(1) *For any object X,* $\quad X \xrightarrow{\mathrm{id}_X} X \quad$ *is a covering.*

(2) *If $(X_i \xrightarrow{x_i} X)_{i \in I}$ is a covering,*
then for any morphism $X' \xrightarrow{x} X$, the associated family
$(X' \times_X X_i \longrightarrow X')_{i \in I}$ *is a covering.*

(3) *If $(X_i \xrightarrow{x_i} X)_{i \in I}$ is a covering and, for any $i \in I$,*
$(X_{i,j} \xrightarrow{x_{i,j}} X_i)_{j \in I_i}$ *is a covering,*
then $(X_{i,j} \xrightarrow{x_{i,j}} X_i \xrightarrow{x_i} X)_{\substack{i \in I \\ j \in I_i}}$ is a covering.

Definition. –

Let $\mathcal{C}$ be a small category which is cartesian
and is endowed with a Grothendieck topology J.
A point of $(\mathcal{C}, J)$, in Grothendieck's sense,
is a mapping

$$\begin{array}{rcl} \rho : \mathcal{C} & \longrightarrow & \mathrm{Set}, \\ \textit{object } X & \longmapsto & \rho(X) = \mathrm{set}, \\ \textit{morphism } (X \xrightarrow{f} Y) & \longmapsto & (\rho(X) \xrightarrow{\rho(f)} \rho(Y)) = \textit{ map}, \end{array}$$

such that:

(F) ρ *is a "functor" , i.e. respects composition of morphisms and identities*

$$\rho(g \circ f) = \rho(g) \circ \rho(f)\,, \qquad \rho(\mathrm{id}_X) = \mathrm{id}_{\rho(X)}\,.$$

(C) ρ *transforms cartesian squares into cartesian squares*

$$\rho(X \times_S S') = \rho(X) \times_{\rho(S)} \rho(S')\,.$$

(C_1) ρ *transforms the terminal objet $1_{\mathcal{C}}$ of $\mathcal{C}$*
into the terminal set $\rho(1_{\mathcal{C}}) = \{\bullet\}$.

(J) ρ *transforms J-covering families* $(X_i \xrightarrow{x_i} X)_{i \in I}$
into globally surjective families $(\rho(X_i) \xrightarrow{\rho(x_i)} \rho(X))_{i \in I}$.

What about the notion of space?

- Starting from the definition of real numbers by approximation, we have defined a notion of point for

  any small cartesian category $\mathcal{C}$
  endowed with a topology J.

- In the particular case where

$$\mathcal{C} = (O, \leq, X, \wedge)$$

  for O a dense subset of

$$O_X = \{\text{open subsets of a sober topological space } X\}$$

  which contains X and is stable by finite intersection,
  and for $J =$ ordinary topology of O,
  we have an identification

$$X = \{\text{points of } (O, J)\}.$$

  This means that the space X is independent from its sketches (O, J) and can be reconstructed from any of them.

Question:

Is it possible to extend this construction to

$$(\mathcal{C}, J) \longmapsto \text{space sketched by } (\mathcal{C}, J)$$

with a well-defined compatible notion of point of such a space?

Recovering the points of a space from its category of "sheaves"

Definition. – *Let X be a topological space.*

(i) *A presheaf on X is a contravariant functor*

$$\begin{array}{rrcl} & P : O_X^{\mathrm{op}} & \longrightarrow & \mathrm{Set}, \\ \textit{open subset} & U & \longmapsto & P(U) = \textit{set}, \\ \textit{inclusion} & (U' \subseteq U) & \longmapsto & (P(U) \to P(U')) = \textit{restriction map}, \end{array}$$

compatible with compositions of inclusions $\quad U'' \subseteq U' \subseteq U.$

(ii) *A presheaf F is a sheaf if, for any covering* $\quad U = \bigcup_{i \in I} U_i \,,$

elements $u \in F(U)$ *correspond one-to-one to families* $\quad u_i \in F(U_i)\,, \quad i \in I\,,$

which are compatible in the sense that, for any $i_1, i_2 \in I$,

$u_{i_1} \in F(U_{i_1})\,,\; u_{i_2} \in F(U_{i_2})$ *have the same restriction in* $F(U_{i_1} \cap U_{i_2})$.

(iii) *The category* $\mathcal{E}_X$ *of sheaves on X is called the "topos" of X.*

Theorem. – *Let X be a "sober" topological space.*

(i) *Elements of X, i.e. points, correspond one-to-one to pairs of adjoint functors*

$$(\mathcal{E}_X \xrightarrow{x^*} \mathrm{Set}\,,\; \mathrm{Set} \xrightarrow{x_*} \mathcal{E}_X)$$

whose left component $x^* : \mathcal{E}_X \to \mathrm{Set}$ *respects terminal objects and cartesian squares.*

(ii) *More generally, for any topological space* X', *continuous maps* $X' \to X$

correspond one-to-one to pairs of adjoint functors $\quad (\mathcal{E}_X \xrightarrow{f^*} \mathcal{E}_{X'}\,,\; \mathcal{E}_{X'} \xrightarrow{f_*} \mathcal{E}_X)$

whose left component $f^* : \mathcal{E}_X \to \mathcal{E}_{X'}$ *respects terminal objects and cartesian squares.*

Generalization to partially ordered sets

Definition. –

Let $(O, \leq, 1, \wedge)$ *be a* *partially ordered set*

with a *biggest element* 1 *and an* *inf operator* $\wedge$.

(i) *A* *presheaf* *on* O *is a* *contravariant functor*

$$\begin{array}{rrcl} & P : O^{\mathrm{op}} & \longrightarrow & \mathrm{Set}, \\ \underline{\textit{element}} & U & \longmapsto & P(U) = \underline{\textit{set}}, \\ \underline{\textit{relation}} & (U' \leq U) & \longmapsto & (P(U) \to P(U')) = \underline{\textit{restriction map}}, \end{array}$$

compatible with compositions *of relations* $\quad U'' \leq U' \leq U$.

(ii) *If* J *is a* Grothendieck topology *on* O,

a *presheaf* F *on* O *is a* J-*sheaf* *if, for any* J-*covering*

$$U = \bigvee_{i \in I} U_i\,,$$

elements $u \in F(U)$ *correspond one-to-one* *to* *families*

$$u_i \in F(U_i)\,, \quad i \in I\,,$$

which are *compatible* *in the sense that, for any* $i_1, i_2 \in I$,

$$u_{i_1} \in F(U_{i_1})\,,\ u_{i_2} \in F(U_{i_2}) \text{ have the } \underline{\text{same restriction}} \text{ in } F(U_{i_1} \cap U_{i_2}).$$

(iii) *The* *category* $\widehat{O}_J$ *of* J-*sheaves* *on* O *is called the "topos" of* (O, J).

Generalization to categories

Definition. – *Consider a small category $\mathcal{C}$.*

(i) *A presheaf on $\mathcal{C}$ is a contravariant functor*

$$\begin{array}{rrcl} & P : \mathcal{C}^{\mathrm{op}} & \longrightarrow & \mathsf{Set}, \\ \textit{object} & X & \longmapsto & P(X) = \textit{set}, \\ \textit{morphism} & (X' \xrightarrow{x} X) & \longmapsto & (P(x) : P(X) \to P(X')) = \textit{restriction map}, \end{array}$$

compatible with compositions of morphisms $\quad X'' \to X' \to X$.

(ii) *If $\mathcal{C}$ is cartesian and J is a Grothendieck topology on $\mathcal{C}$, a presheaf on $\mathcal{C}$ is a J-sheaf if, for any J-covering*

$$X_i \xrightarrow{x_i} X\,, \quad i \in I\,,$$

elements $u \in F(X)$ correspond one-to-one to families

$$u_i \in F(X_i)\,, \quad i \in I\,,$$

which are compatible in the sense that, for any $i_1, i_2 \in I$,

$$u_{i_1} \in F(X_{i_1})\,,\ u_{i_2} \in F(X_{i_2}) \text{ have the same restriction in } F(X_{i_1} \times_X X_{i_2}).$$

(iii) *The category $\widehat{\mathcal{C}}_J$ of sheaves on $(\mathcal{C}, J)$ is called the "topos" of $(\mathcal{C}, J)$.*

Definition. – *A topos $\mathcal{E}$ is a category which admits an equivalence*

$$\mathcal{E} \xrightarrow{\sim} \widehat{\mathcal{C}}_J$$

to the topos of sheaves on some $(\mathcal{C}, J)$.

Independence of sketches

Proposition (Grothendieck). –

Let $\mathcal{C}$ be a small (cartesian) category $\mathcal{C}$ endowed with a topology J.
Let

$$\mathcal{C}' \hookrightarrow \mathcal{C}$$

be a full (cartesian) subcategory which is "dense"

in the sense that any object X of $\mathcal{C}$ admits J-coverings

$$(X_i \xrightarrow{x_i} X)_{i \in I}$$

by objects X_i of $\mathcal{C}'$.
Let J' be the topology of $\mathcal{C}'$ induced by the topology J of $\mathcal{C}$.

Then there is a canonical equivalence of toposes

$$\widehat{\mathcal{C}}_J \xrightarrow{\sim} \widehat{\mathcal{C}}'_{J'} \,.$$

Corollary. –
Let X be a topological space.

Let $O \subseteq O_X$ be a dense subset of the set $O_X = \{$open subsets of $X\}$
(which contains X and is stable under finite intersections).
Then there is an equivalence of toposes

$$\mathcal{E}_X \xrightarrow{\sim} \widehat{O}_J$$

if J is the ordinary topology on O.

Points of toposes

Definition. –

(i) *A point of a topos* $\mathcal{E}$ *is a pair of adjoint functors*

$$(\mathcal{E} \xrightarrow{x^*} \mathrm{Set}\,,\ \mathrm{Set} \xrightarrow{x_*} \mathcal{E})$$

whose left component $x^* : \mathcal{E} \to \mathrm{Set}$ *respects terminal objects and cartesian squares.*

(ii) *A morphism of toposes* $\mathcal{E}' \xrightarrow{f} \mathcal{E}$

(or a mobile point of $\mathcal{E}$ *parametrized by* $\mathcal{E}'$*)*

is a pair of adjoint functors

$$(\mathcal{E} \xrightarrow{f^*} \mathcal{E}'\,,\ \mathcal{E}' \xrightarrow{f_*} \mathcal{E})$$

whose left component $f^* : \mathcal{E} \to \mathcal{E}'$ *respects terminal objects and cartesian squares.*

Remark:

This definition is motivated by the fact that, for

a sober topological space X and a topological space X',

continuous maps

$$X' \longrightarrow X$$

correspond one-to-one to topos morphisms

$$\mathcal{E}_{X'} \longrightarrow \mathcal{E}_X\,.$$

Approximation of points

Theorem (Grothendieck, Diaconescu). –
Let $\mathcal{E}$ be a topos sketched as

$$\mathcal{E} \xrightarrow{\sim} \widehat{\mathcal{C}}_J$$

for a small (cartesian) category $\mathcal{C}$

endowed with a topology J.

Then points of $\mathcal{E}$ [resp. topos morphisms $\mathcal{E}' \to \mathcal{E}$]

correspond one-to-one to functors

$$\rho : \mathcal{C} \longrightarrow \mathrm{Set} \quad \textit{[resp.} \;\; \rho : \mathcal{C} \longrightarrow \mathcal{E}'\textit{]}$$

which verify the following conditions:

(C) *The functor ρ respects cartesian squares.*

(C_1) *It transforms the terminal object 1 of $\mathcal{C}$*

into the terminal object $\{\bullet\}$ of Set [resp. $1_{\mathcal{E}'}$ of $\mathcal{E}'$].

(J) *The functor ρ transforms J-coverings*

$$(X_i \xrightarrow{x_i} X)_{i \in I}$$

into families of maps of Set [resp. of morphisms of $\mathcal{E}'$]

which are globally surjective [resp. globally epimorphic].