Computing with subspaces

Laurent LAFFORGUE

(Centre Lagrange, Huawei Technologies France)

(Westlake Fellowship invited professor)

June, 2026

Westlake University

Westlake Distinguished Lecture

The relational point of view in mathematics

Principle: Every mathematical object must be considered
within the environment of objects of its type.

Key phenomenon (which is a basic theorem):

Considered within the environment of objects of its type,
every mathematical object is characterized by
the network of its relations with objects of that type.

Formalization: the notion of a "category"

A category (or "mathematical country") consists of

- an oriented graph consisting in
  - vertices (or "mathematical cities"),
  - arrows (or "mathematical routes")

    $$\bullet \longrightarrow \bullet$$

    going from a "source" vertex to a "target" vertex,
- an associative law of composition of routes

$$\left(X \xrightarrow{f} Y \xrightarrow{g} Z\right) \longmapsto \left(X \xrightarrow{g \circ f} Z\right).$$

Some general ideas about topoi

- The notion of topos was introduced by Grothendieck around 1960, first by a constructive definition:

  A topos is a category that can be constructed
  as the category of "sheaves" on
  a "small category" endowed with a "topology".

- Later, Giraud gave an equivalent axiomatic definition:

  A topos is a category
  that satisfies the same explicit list of "good constructive properties"
  as the category of sets.

- Grothendieck defined a notion of geometric morphism between toposes

$$\mathcal{E}' \longrightarrow \mathcal{E}$$

  and an associative law of composition of such morphisms

$$\left(\mathcal{E}'' \xrightarrow{f} \mathcal{E}' \xrightarrow{g} \mathcal{E}\right) \longmapsto \left(\mathcal{E}'' \xrightarrow{g \circ f} \mathcal{E}\right).$$

- Toposes form a category.
- The geometric morphisms $\mathcal{E}' \to \mathcal{E}$ from a topos $\mathcal{E}'$ to a topos $\mathcal{E}$ form a category $\mathcal{G}eom(\mathcal{E}', \mathcal{E})$.

Generalization of the notions of space and point

- Every topological space X can be viewed as a topos $\mathcal{E}_X$, but there are many more toposes than topological spaces.
- For all topological spaces X, Y, there is a correspondence

$$\left\{ \frac{\text{continuous maps}}{X \longrightarrow Y} \right\} \quad \longleftrightarrow \quad \left\{ \frac{\text{topos-theoretic geometric morphisms}}{\mathcal{E}_X \longrightarrow \mathcal{E}_Y} \right\}.$$

- In particular, the space reduced to one point

$$\{\bullet\}$$

can be viewed as a topos $\mathcal{E}_{\{\bullet\}}$,
and one may call "points" of a topos $\mathcal{E}$ the topos-theoretic geometric morphisms

$$\mathcal{E}_{\{\bullet\}} \longrightarrow \mathcal{E}.$$

- More generally, since continuous maps

$$S \to X \qquad \text{from a space } S \text{ to a space } X$$

may be called "moving points" of X parametrized by S,
the topos-theoretic geometric morphisms

$$\mathcal{E}' \longrightarrow \mathcal{E}$$

may be called the "moving points" of the topos $\mathcal{E}$ parametrized by a topos $\mathcal{E}'$.

From the syntax of logical systems or theories to their semantics

- A logical system, or ("first-order geometric") theory $\mathbb{T}$ consists of

  – a vocabulary (made up of "object names", "function names", "relation names"),

  – a list of grammar rules, called "axioms", which have the form of implications

  $$\varphi \vdash \psi$$

  between "geometric formulas" written in this "vocabulary".

- Every such theory $\mathbb{T}$, viewed as a "syntax", has a "semantics" which consists of

  – the category $\mathbb{T}\text{-mod}\,(\mathrm{Set})$ of its set-based models,

  – more generally, the categories $\mathbb{T}\text{-mod}\,(\mathcal{E})$

  of its models in topos environments $\mathcal{E}$,

  – the transformations of "change of environment"

  $$\mathbb{T}\text{-mod}\,(\mathcal{E}) \xrightarrow{f^{-1}} \mathbb{T}\text{-mod}\,(\mathcal{E}')$$

  defined by the topos-theoretic geometric morphisms

  $$f : \mathcal{E}' \longrightarrow \mathcal{E}\,.$$

The topos-theoretic incarnation of the semantics of theories

Around 1975, Makkai and Reyes,
building on Grothendieck, Hakim, Lawvere, Joyal, $\cdots$ proved:

Theorem. – *For every (first-order geometric) theory* $\mathbb{T}$*,*
there exists a topos $\mathcal{E}_{\mathbb{T}}$ *(unique up to equivalence) such that*

- *the category of set-based models* $\mathbb{T}$-mod (Set) *is identified with the category of points*

$$\mathrm{pt}(\mathcal{E}_{\mathbb{T}}) = \mathcal{G}eom(\mathcal{E}_{\{\bullet\}}, \mathcal{E}_{\mathbb{T}}) = \{\mathcal{E}_{\{\bullet\}} \to \mathcal{E}_{\mathbb{T}}\},$$

- *more generally, for every topos* $\mathcal{E}$*, the category*

$$\mathbb{T}\text{-mod}(\mathcal{E}) \quad \textit{of models of } \mathbb{T} \textit{ in the environment } \mathcal{E}$$

  is identified with the category of moving points of $\mathcal{E}_{\mathbb{T}}$ *parametrized by* $\mathcal{E}$

$$\mathcal{G}eom(\mathcal{E}, \mathcal{E}_{\mathbb{T}}) = \{\mathcal{E} \to \mathcal{E}_{\mathbb{T}}\},$$

- *for every topos-theoretic geometric morphism* $\mathcal{E}' \xrightarrow{f} \mathcal{E}$*, the change-of-environment transformation*

$$f^{-1} : \mathbb{T}\text{-mod}(\mathcal{E}) \longrightarrow \mathbb{T}\text{-mod}(\mathcal{E}')$$

  corresponds to the law of composition with f

$$\left(\mathcal{E} \xrightarrow{p} \mathcal{E}_{\mathbb{T}}\right) \longmapsto \left(\mathcal{E}' \xrightarrow{p \circ f} \mathcal{E}_{\mathbb{T}}\right).$$

A notion of subtopos with the properties of subspaces

Grothendieck defined a notion of subtopos of a topos $\mathcal{E}$

$$\mathcal{E}' \hookrightarrow \mathcal{E}$$

such that:

- For every topos $\mathcal{E}$, its subtoposes form a set partially ordered by the factorization relation $\supseteq$

$$(\mathcal{E}' \hookrightarrow \mathcal{E}) \supseteq (\mathcal{E}'' \hookrightarrow \mathcal{E}) \iff \begin{tikzcd} \mathcal{E}'' \arrow[rr, hook] \arrow[dr, hook] & & \mathcal{E}' \arrow[dl, hook] \\ & \mathcal{E} & \end{tikzcd}$$

- Every family of subtoposes $\quad \mathcal{E}_i \hookrightarrow \mathcal{E}\,, \quad i \in I\,,$

  has a union $\bigvee\limits_{i\in I} \mathcal{E}_i$ characterized by $\quad \mathcal{E}' \supseteq \bigvee\limits_{i\in I} \mathcal{E}_i \iff \mathcal{E}' \supseteq \mathcal{E}_i\,, \quad \forall\, i \in I\,,$

  and an intersection $\bigwedge\limits_{i\in I} \mathcal{E}_i$ characterized by $\quad \bigwedge\limits_{i\in I} \mathcal{E}_i \supseteq \mathcal{E}' \iff \mathcal{E}_i \supseteq \mathcal{E}'\,, \quad \forall\, i \in I\,.$

- Every pair of subtoposes $\quad \mathcal{E}_1 \hookrightarrow \mathcal{E} \quad$ et $\quad \mathcal{E}_2 \hookrightarrow \mathcal{E}$

  defines a "difference" subtopos $\mathcal{E}_1 \backslash \mathcal{E}_2$ characterized by $\quad \mathcal{E}' \supseteq \mathcal{E}_1 \backslash \mathcal{E}_2 \iff \mathcal{E}' \vee \mathcal{E}_2 \supseteq \mathcal{E}_1\,.$

- Every topos-theoretic geometric morphism $\quad \mathcal{E}' \xrightarrow{f} \mathcal{E} \quad$ factors through a smallest subtopos $\quad \mathcal{E}' \longrightarrow \mathrm{Im}(f) \hookrightarrow \mathcal{E}$ called the image $\quad \mathrm{Im}(f) \quad$ of f.

External operations on subtoposes

Proposition. – *Every topos-theoretic geometric morphism*

defines $$f : \mathcal{E}' \longrightarrow \mathcal{E}$$

(i) *a direct image operator on subtoposes*

$$\exists_f = f_* : \{\textit{subtoposes of } \mathcal{E}'\} \longrightarrow \{\textit{subtoposes of } \mathcal{E}\},$$

$$(\mathcal{E}_1' \hookrightarrow \mathcal{E}') \longmapsto f_*\mathcal{E}_1' = \mathrm{Im}(\mathcal{E}_1' \hookrightarrow \mathcal{E}' \xrightarrow{f} \mathcal{E}),$$

(ii) *an inverse image operator on subtoposes*

$$f^{-1} : \{\textit{subtoposes of } \mathcal{E}\} \longrightarrow \{\textit{subtoposes of } \mathcal{E}'\},$$

$$(\mathcal{E}_1 \hookrightarrow \mathcal{E}) \longmapsto (f^{-1}\mathcal{E}_1 \hookrightarrow \mathcal{E}')$$

characterized by $f^{-1}\mathcal{E}_1 \supseteq \mathcal{E}_1' \Leftrightarrow \mathcal{E}_1 \supseteq f_*\mathcal{E}'_1$.

Remarks:

(i) f_* preserves arbitrary unions, and f^{-1} preserves arbitrary intersections.

(ii) (O.C., L.L.) f^{-1} also preserves finite unions.

(iii) (O.C., L.L.) If f is "simply connected",

f^{-1} preserves arbitrary unions, and there exists an operator

$$\forall_f = f_! : \{\text{subtoposes of } \mathcal{E}'\} \longrightarrow \{\text{subtoposes of } \mathcal{E}\},$$

$$(\mathcal{E}_1' \hookrightarrow \mathcal{E}') \longmapsto (f_!\mathcal{E}_1' \hookrightarrow \mathcal{E})$$

characterized by $\quad f_!\mathcal{E}_1' \supseteq \mathcal{E}_1 \Longleftrightarrow \mathcal{E}_1' \supseteq f^{-1}\mathcal{E}_1$.

The notion of Grothendieck topology

Recall.– A "topos" is a category $\mathcal{E}$ that can be presented (up to equivalence) as a "category of sheaves"

$$\mathcal{E} \cong \widehat{\mathcal{C}}_J$$

on a "small category" $\mathcal{C}$ endowed with a topology J.

Definition. – *A topology J on a category $\mathcal{C}$ consists of a notion of covering of the vertices X of $\mathcal{C}$ by families of arrows with target X*

$$\left(X_i \xrightarrow{x_i} X\right)_{i\in I}$$

which satisfies the following three properties:

(Composition) For every covering $\left(X_i \xrightarrow{x_i} X\right)_{i\in I}$
and all coverings $\left(X_{i,j} \xrightarrow{x_{i,j}} X_i\right)_{j\in I_i}$,
the composites $\left(X_{i,j} \xrightarrow{x_{i,j}} X_i \to X\right)_{i\in I, j\in I_i}$ *form a covering.*

(Induction) Every arrow $X \to Y$ transforms coverings of Y into coverings of X.

(Factorization) If every element $X_i \xrightarrow{x_i} X$ *of a covering* $\left(X_i \xrightarrow{x_i} X\right)_{i\in I}$ *factors through an arrow* $X_{i'} \xrightarrow{x_{i'}} X$, *then the* $X_{i'} \xrightarrow{x_{i'}} X$ *form a covering of X.*

Subtoposes and topologies

Proposition (Grothendieck, SGA 4). –
For every topos $\mathcal{E}$ presented as a category of sheaves

$$\mathcal{E} \cong \widehat{\mathcal{C}}_J$$

on a small category $\mathcal{C}$ endowed with a topology J, we have:

(i) *Every topology J' on $\mathcal{C}$ that refines J*
(in the sense that every covering for J is a covering for J')
defines a subtopos

$$\widehat{\mathcal{C}}_{J'} \hookrightarrow \widehat{\mathcal{C}}_J \cong \mathcal{E}.$$

(ii) *Conversely, every subtopos*

$$\mathcal{E}' \longhookrightarrow \mathcal{E} \cong \widehat{\mathcal{C}}_J$$

is of the form $\mathcal{E}' \cong \widehat{\mathcal{C}}_{J'}$ for a unique topology J' that refines J.

Consequence:
The operations on subtoposes (union, intersection, difference,
direct image f_* by a topos-theoretic geometric morphism $\mathcal{E}' \xrightarrow{f} \mathcal{E}$, inverse image $f^{-1}, \cdots$)
correspond to operations on topologies.

Observation:
A topology is given by a family of coverings that generate it.

Subspaces and subtoposes

Recall. – *If X is a topological space, its topos $\mathcal{E}_X$ is the topos of sheaves*

$$\mathcal{E}_X = \widehat{(\mathcal{C}_X)}_{J_X}$$

on $\mathcal{C}_X =$ the category of open sets of X,
for $J_X =$ the usual notion of covering

$$(U_i \subseteq U)_{i\in I} \quad \text{characterized by } \bigcup_{i\in I} U_i = U.$$

Observation. – *Every subspace $Y \hookrightarrow X$ defines a subtopos*

$$\mathcal{E}_Y = \widehat{(\mathcal{C}_X)}_{J_Y} \longrightarrow \widehat{(\mathcal{C}_X)}_{J_X}$$

defined by the topology J_Y on $\mathcal{C}_X$ for which
a family $(U_i \subseteq U)_{i\in I}$ is a covering
if and only if $\bigcup_{i\in I} (U_i \cap Y) = U \cap Y$.

Observation: There are many more topologies on $\mathcal{C}_X$, and therefore many more subtoposes of $\mathcal{E}_X$, than subspaces $Y \hookrightarrow X$.

Example: If X is endowed with a measure μ,
$\mathcal{C}_X$ can be endowed with the topology J_μ for which
a family $(U_i \subset U)_{i\in I}$ is a covering if it has a countable subfamily

$(U_{i_n} \subset U)_{n\in\mathbb{N}}$ such that $\mu\left(U - \bigcup_{n\in\mathbb{N}} U_{i_n}\right) = 0.$

Caramello's duality theorem

Theorem. –
Consider a (first-order geometric) theory $\mathbb{T}$ *written in a formal language* Σ*, with its associated topos*

$$\mathcal{E}_{\mathbb{T}}.$$

Then:

(i) *Every theory* $\mathbb{T}'$ *that is a "quotient" of* $\mathbb{T}$
that is, is written in the same language Σ
and is deduced from $\mathbb{T}$ *by adding extra axioms,*
defines a subtopos

$$\mathcal{E}_{\mathbb{T}'} \hookrightarrow \mathcal{E}_{\mathbb{T}}.$$

(ii) *Two quotient theories* $\mathbb{T}_1$ *and* $\mathbb{T}_2$ *of* $\mathbb{T}$
define the same subtopos of $\mathcal{E}_{\mathbb{T}}$
if and only if the axioms of one
are provable from the axioms of the other.

(iii) *Every subtopos*

$$\mathcal{E}' \hookrightarrow \mathcal{E}_{\mathbb{T}}$$

is of the form $\mathcal{E}' = \mathcal{E}_{\mathbb{T}'}$ *for quotient theories* $\mathbb{T}'$ *of* $\mathbb{T}$.

Consequence:
All provability problems in first-order geometric logic
are constructively equivalent to problems of generation of topologies.

Application to syntactic learning

Proposition. –

Consider a model M in a topos $\mathcal{E}$
of a (first-order geometric) theory $\mathbb{T}$.
Consider the topos-theoretic geometric morphism corresponding to M

$$m : \mathcal{E} \longrightarrow \mathcal{E}_{\mathbb{T}}$$

and then its image

$$\mathrm{Im}(m) \hookrightarrow \mathcal{E}_{\mathbb{T}}$$

necessarily defined by a quotient theory $\mathbb{T}'$ of $\mathbb{T}$, avec

$$\mathrm{Im}(m) = \mathcal{E}_{\mathbb{T}'} \hookrightarrow \mathcal{E}_{\mathbb{T}}.$$

Then an implication between geometric formulas

$$\varphi \vdash \psi$$

is provable in $\mathbb{T}'$
if and only if it is satisfied by the model M.

Remark:

Thus, $\mathbb{T}'$ is the syntactic "description"

of the model M in the vocabulary Σ of $\mathbb{T}$.

Arithmetic of theories

- For every topos $\mathcal{E}$
  that incarnates the semantics of a theory $\mathbb{T}$ in the sense that
  $$\mathcal{E} \cong \mathcal{E}_{\mathbb{T}},$$
  the internal operations on subtoposes of $\mathcal{E}$
  (union, intersection, difference)
  correspond to operations on the quotient theories of $\mathbb{T}$,
  that is, on the systems of axioms written in the language of $\mathbb{T}$.
- For every topos-theoretic geometric morphism
  $$f : \mathcal{E}' \longrightarrow \mathcal{E}$$
  which connects two toposes incarnating the semantics of two theories $\mathbb{T}$ and $\mathbb{T}'$,
  in the sense that $\mathcal{E}' \cong \mathcal{E}_{\mathbb{T}'}\,, \quad \mathcal{E} \cong \mathcal{E}_{\mathbb{T}},$
  the external operations induced by f

  f_*, f^{-1} and possibly $f_!$

  correspond to operations on the quotient theories of $\mathbb{T}$ and $\mathbb{T}'$.

Reduction to "locally computable" categories

Proposition (O.C., L.L.). –
*Every topos $\mathcal{E}$ can be presented, in many ways,
as a topos of sheaves for a topology J*

$$\mathcal{E} \cong \widehat{\mathcal{C}}_J$$

on categories $\mathcal{C}$ associated with a vocabulary

$$\Sigma = \{\textit{object names, relation names}\}$$

and which are "locally computable" in the sense that

- *their objects are indexed by finite data
of an explicit type which depends only on the vocabulary Σ,*
- *the arrows between two such objects
always form a computable finite set,*
- *the law of composition of arrows is explicit and easily computable,*
- *every arrow of $\mathcal{C}$*

$$X \longrightarrow Y$$

*transforms every arrow $Y' \to Y$ with target Y into an arrow $X' \to X$ with target X,
by an easily computable procedure.*

Consequence: All problems are constructively reduced
to problems of generation of topologies on such categories.