Galois connections

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A very general context

Definition. –

Let C and S be two sets or, more generally, two classes.

A relation between C and S is a subset or subclass

$$R \hookrightarrow C \times S\,.$$

A familiar family of examples:

Consider a set C endowed with some structures.

Consider a set S of "symmetries" of C, i.e. bijections

$$\sigma : C \xrightarrow{\sim} C$$

which preserve the structures of C.

(NB: Of course, S generates a group of similar symmetries.)

Then a natural relation between C and S is

$$R = \{(x, \sigma) \in C \times S \mid \sigma(x) = x\}\,.$$

The Galois case:

C is a field,

S is a group of symmetries of C respecting the field structure.

A couple of induced natural maps

Definition. –

Consider an arbitrary relation

$$R \hookrightarrow C \times S\,.$$

Then R defines a couple of maps

$$\left\{ \begin{array}{c} \underline{\text{subsets}} \text{ or } \underline{\text{subclasses}} \\ \text{of } C \end{array} \right\} \underset{G_R}{\overset{R_F}{\rightleftarrows}} \left\{ \begin{array}{c} \underline{\text{subsets}} \text{ or } \underline{\text{subclasses}} \\ \text{of } S \end{array} \right\}$$

by the formulas

$$\begin{array}{ccc} (K \subseteq C) & \longmapsto & (F_R(K) \subseteq S) \\ & & \| \\ & & \{\sigma \in S \mid (x, \sigma) \in R\,,\ \forall\, x \in K\}\,, \end{array}$$

$$\begin{array}{ccc} (H \subseteq S) & \longmapsto & (G_R(H) \subseteq C) \\ & & \| \\ & & \{x \in C \mid (x, \sigma) \in R\,,\ \forall\, \sigma \in H\}\,. \end{array}$$

Remark:

The formulas are deduced from each other

by exchanging C and S.

Simple essential formal properties

Lemma. –
Consider an arbitrary relation $R \hookrightarrow C \times S$
and the associated couple of maps

$$\{K \subseteq C\} \underset{G_R}{\overset{R_F}{\rightleftarrows}} \{H \subseteq S\}\,.$$

Then:

(i) *The two maps F_R and G_R reverse inclusions in the sense that*

$$\begin{aligned} K' \subseteq K \quad &\Longrightarrow \quad F_R(K') \supseteq F_R(K)\,, \\ H' \supseteq H \quad &\Longrightarrow \quad G_R(H') \subseteq G_R(H)\,. \end{aligned}$$

(ii) *They are "adjoint" in the sense that, for any*

$$K \subseteq C \quad \text{and} \quad H \subseteq S\,,$$

there is an equivalence

$$F_R(K) \supseteq H \Leftrightarrow K \subseteq G_R(H)\,.$$

Proof:

(i) is obvious.

(ii) Both relations $F_R(K) \supseteq H$ and $K \subseteq G_R(H)$ are equivalent to

$K \times H \subseteq R$ as subsets or subclasses of $C \times S$.

Reminder on categories

Definition. –
A "category" (or "mathematical country") consists in

- *"objects" (or "mathematical cities")* $X, Y, Z, \cdots$
- *"morphisms" (or "mathematical itineraries") relating pairs of objects (or mathematical cities)*

$$X \xrightarrow{f} Y\,,$$

- *a composition law of itineraries*

$$(X \xrightarrow{f} Y \xrightarrow{g} Z) \longmapsto (X \xrightarrow{g\circ f} Z)$$

which

$$\begin{cases} - & \text{is associative, meaning that for } X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} W, \\ & \qquad h \circ (g \circ f) = (h \circ g) \circ f\,, \\ - & \text{has units } \mathrm{id}_X : X \to X \text{ associated to cities } X \text{ verifying} \\ & \mathrm{id}_X \circ f = f \quad \text{for} \quad X' \xrightarrow{f} X\,, \quad g \circ \mathrm{id}_X = g \quad \text{for} \quad X \xrightarrow{g} Y\,. \end{cases}$$

Example:
Any ordered set of class $(O, \leq)$ can be seen as a category
where elements U, V are related by

$$\begin{cases} \bullet & \text{a unique morphism } U \to V \text{ if } U \leq V, \\ \bullet & \text{no morphism at all if } U \leq V \text{ is not verified.} \end{cases}$$

Reminder on functors

Definition. –
Let $\mathcal{C}, \mathcal{D}$ be categories (or "mathematical countries").
A functor (or a "mathematical international relationship")
is a double mapping

$$\begin{array}{rcrcl} F & : & \{\text{cities of } \mathcal{C}\} & \longrightarrow & \{\text{cities of } \mathcal{D}\} \\ & & X & \longmapsto & F(X)\,, \\ & & \{\text{itineraries of } \mathcal{C}\} & \longrightarrow & \{\text{itineraries of } \mathcal{D}\}\,, \\ & & (X \xrightarrow{f} Y) & \longmapsto & (F(X) \xrightarrow{F(f)} F(Y)\,, \end{array}$$

such that

- *F respects compositions $X \xrightarrow{f} Y \xrightarrow{g} Z$, meaning*
$$F(g \circ f) = F(g) \circ F(f)\,,$$
- *F respects units $X \xrightarrow{\mathrm{id}_X} X$, meaning*
$$F(\mathrm{id}_X) = \mathrm{id}_{F(X)}\,.$$

Example:
A map between ordered sets or classes seen as categories

$$F : (O, \leq) \longrightarrow (O', \geq)$$

is a functor if and only if it respects order relations, i.e.

$$K \leq K' \implies F(K) \geq F(K')\quad.$$

Reminder on adjoint functors

Definition. –
A pair (F,G) of functors between categories

$$\mathcal{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{D}$$

is called "adjoint" if, for any objects (= cities)

$$X \text{ of } \mathcal{C} \quad , \quad Y \text{ of } \mathcal{D} \quad ,$$

itineraries $\quad F(X) \to Y \quad$ *in* $\mathcal{D}$
have a "natural" one-to-one correspondence with
itineraries $\quad X \to G(Y) \quad$ *in* $\mathcal{C}$.

Remark:
In particular, the units $F(X) \xrightarrow{\text{id}} F(X)$ in $\mathcal{D}$ correspond to itineraries

$$X \longrightarrow G \circ F(X) \quad \text{in } \mathcal{C},$$

and the units $G(Y) \xrightarrow{\text{id}} G(Y)$ in $\mathcal{C}$ correspond to itineraries

$$F \circ G(Y) \longrightarrow Y \quad \text{in } \mathcal{D}.$$

The particular case of ordered sets or classes:
A pair of maps which respect order relations

$$(O, \leq) \underset{G}{\overset{F}{\rightleftarrows}} (O', \geq)$$

is "adjoint" if and only if, for any $K \in O$, $H \in O'$,

$$F(K) \geq H \iff K \leq G(H)\,.$$

Fixed points

Proposition. –

Consider a pair (F, G) *of adjoint functors between categories*

$$\mathcal{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{D}\,.$$

Let $\mathcal{C}'$ *[resp.* $\mathcal{D}'$*] be the full subcategory of* $\mathcal{C}$ *[resp.* $\mathcal{D}$*]*

consisting of objects (= cities)

$$X \qquad \text{[resp. } Y\text{]}$$

such that the canonical morphism

$$X \longrightarrow G \circ F(X) \quad \textit{corresponding to} \quad F(X) \xrightarrow{\mathrm{id}} F(X)$$

[resp.

$$F \circ G(Y) \longrightarrow Y \quad \textit{corresponding to} \quad G(Y) \xrightarrow{\mathrm{id}} G(Y)\text{]}$$

is an isomorphism [i.e. has an inverse].

Then F and G induce "equivalences"

$$\mathcal{C}' \underset{\underset{G}{\sim}}{\overset{\overset{F}{\sim}}{\rightleftarrows}} \mathcal{D}'$$

which are inverse to each other.

Corollary. –
Consider an adjoint pair (F, G) of maps respecting order relations

$$(O, \leq) \underset{G}{\overset{F}{\rightleftarrows}} (O', \geq)\,.$$

Then F and G induce two bijections inverse to each other
between the subsets or subclasses of fixed points

$$\{K \in O \mid K = G \circ F(K)\} \underset{\underset{G}{\sim}}{\overset{\overset{F}{\sim}}{\rightleftarrows}} \{H \in O' \mid F \circ G(H) = H\}\,.$$

Lemma. –
In that case, an element $K \in O$ [resp. $H \in O'$]
is a fixed point if and only if it is an image, i.e.

$$\exists\, H \in O'\,,\ K = G(H) \qquad \textit{[resp.} \quad \exists\, K \in O\,,\ H = F(K)\textit{].}$$

Proof:
The condition is clearly necessary.
It is sufficient as, if $K = G(H)$, the relations

$$K \leq G \circ F(K)\,, \quad F \circ G(H) \geq H \quad \text{i.e.} \quad F(K) \geq H$$

imply $\quad G \circ F(K) \leq G(H) = K \leq G \circ F(K)$

which means $\quad K = G \circ F(K)\,.$

Generated fixed points

Corollary. –
Consider an adjoint pair (F, G) of maps respecting order relations

$$(O, \leq) \underset{G}{\overset{F}{\rightleftarrows}} (O', \geq)\,.$$

Then, for any $K \in O$ [resp. $H \in O'$], one has

$$K \leq G \circ F(K) \qquad [\textit{resp.} \quad F \circ G(H) \geq H]$$

and $G \circ F(K)$ [resp. $F \circ G(H)$]

is the smallest fixed point which is "bigger" than K [resp. H].

Application:
This applies to any relation $R \hookrightarrow C \times S$
and the associated adjoint pair of order respecting maps:

$$\begin{array}{ccc} (O_C, \subseteq) & \underset{G_R}{\overset{F_R}{\rightleftarrows}} & (O_S, \supseteq) \\ \| & & \| \\ \left\{ \begin{array}{c} \text{subsets or subclasses} \\ \text{of } C \end{array} \right\} & & \left\{ \begin{array}{c} \text{subsets or subclasses} \\ \text{of } S \end{array} \right\} \end{array}$$

For any $(K \subseteq C) \in O_C$ [resp. $(H \subseteq S) \in O_S$], we have

$$K \subseteq G_R \circ F_R(K) \qquad [\text{resp. } \; F_R \circ G_R(H) \supseteq H]$$

and $G_R \circ F_R(K)$ [resp. $F_R \circ G_R(H)$]

is the fixed point generated by $K \subseteq C$ [resp. by $H \subseteq S$].

Generation of dualities

Start with a relation

$$R \hookrightarrow C \times S$$

and the associated pair of order respecting maps

$$(O_C, \subseteq) \quad \underset{G_R}{\overset{F_R}{\rightleftarrows}} \quad (O_S, \supseteq)\,.$$

What is to be done:
Identify the fixed points

$$\begin{aligned} O'_C &= \{(K \subseteq C) \in O_C\,,\ G_R \circ F_R(K) = K\}, \\ O'_S &= \{(H \subseteq S) \in O_S\,,\ F_R \circ G_R(H) = H\}. \end{aligned}$$

Reminder:
An element is a fixed point
if and only if it is an image.

Induced dualities. –

After identifying O'_C and O'_S
one gets a "duality" consisting in a pair
of inclusion reversing bijections inverse to each other

$$(O'_C, \subseteq) \underset{\underset{\sim}{G_R}}{\overset{\overset{F_R}{\sim}}{\rightleftarrows}} (O'_S, \supseteq)\,.$$

Galois duality

Start with C = field,
S = finite subgroup of the group of symmetries of C,
and $R \hookrightarrow C \times S$ defined as

$$R = \{(x, \sigma) \in C \times S \mid \sigma(x) = x\},$$

inducing

$$(O_C, \subseteq) \underset{G_R}{\overset{F_R}{\rightleftarrows}} (O_S, \supseteq).$$

Lemma. –

(i) *Any image of F_R is a subgroup of S.*

(ii) *Any image of G_R is a subfield which contains*

$$G_R(S) = C_S = \{x \in C \mid \sigma(x) = x\,,\ \forall\, \sigma \in S\}.$$

Galois' duality is obtained by proving:

Theorem. –

(i) *The fixed points of $F_R \circ G_R$ in O_S are exactly the subgroups of S.*

(ii) *The fixed points of $G_R \circ F_R$ in O_C are exactly the intermediate subfields $C_S \subseteq K \subseteq C$.*

Historical overview:
Marcel Erné, "Adjunctions and Galois connections: Origin, History and Developement"

Reminder on presheaves and presieves

Definition. – *Let $\mathcal{C}$ be a small category (= "mathematical country").*

(i) *A presheaf on $\mathcal{C}$ is a "contravariant functor"*

$$\begin{array}{rcl} P : \mathcal{C}^{\mathrm{op}} & \longrightarrow & \mathrm{Set}, \\ \text{city } X & \longmapsto & P(X) = \text{set}, \\ \text{itinerary } (X' \xrightarrow{x} X) & \longmapsto & (P(X) \xrightarrow{P(x)} P(X')) = \text{restriction map} \end{array}$$

transforming composites $X'' \xrightarrow{x'} X' \xrightarrow{x} X$ into composites $P(X) \xrightarrow{P(x)} P(X') \xrightarrow{P(x')} P(X'')$.

(ii) *A morphism of presheaves $P \xrightarrow{u} Q$ is a family of maps $P(X) \xrightarrow{u_X} Q(X)$ indexed by cities X of $\mathcal{C}$ which are "compatible" with itineraries $X' \xrightarrow{x} X$ of $\mathcal{C}$, i.e. define commutative squares:*

$$\begin{tikzcd} P(X) \arrow[r, "u_X"] \arrow[d, "P(x)"'] & Q(X) \arrow[d, "Q(x)"] \\ P(X') \arrow[r, "u_{X'}"] & Q(X') \end{tikzcd}$$

Remark: Presheaves on $\mathcal{C}$ make up a category $\widehat{\mathcal{C}}$.

Definition. –

(i) *A presieve on a city X of $\mathcal{C}$ is a family of itineraries $(X_i \xrightarrow{x_i} X)_{i \in I}$ which all go to X.*

(ii) *A "pull-back" of a presieve $(X_i \xrightarrow{x_i} X)_{i \in I}$ by a morphism $X' \xrightarrow{x} X$ is a presieve $(X'_j \xrightarrow{x'_j} X')_{j \in I'}$ such that a morphism $X'' \xrightarrow{x'} X'$ factorizes through at least one $X'_j \xrightarrow{x'_j} X'$ if and only if $x \circ x' : X'' \to X' \to X$ factorizes through at least one $X_i \xrightarrow{x_i} X$.*

The glueing property

Observation. –
If $(X_i \xrightarrow{x_i} X)_{i\in I}$ is a presieve on a city X of a category $\mathcal{C}$, P is a presheaf on $\mathcal{C}$, any element $p \in P(X)$ defines by restriction a family of elements

$$P(x_i)(p) = p_i \in P(X_i)\,, \quad i \in I\,,$$

which is "coherent" in the sense that for any commutative square

$$\begin{tikzcd} U \arrow[r] \arrow[d] & X_{i_2} \arrow[d, "x_{i_2}"] \\ X_{i_1} \arrow[r, "x_{i_1}"'] & X \end{tikzcd} \qquad \textit{with } \; i_1, i_2 \in I \quad ,$$

$p_{i_1} \in P(X_{i_1})$ and $p_{i_2} \in P(X_{i_2})$ have the same restriction in $P(U)$.

Definition. –
Let $\mathcal{X} = (X_i \xrightarrow{x_i} X)_{i\in I}$ be a presieve on a city X of $\mathcal{C}$ and P be a presheaf on $\mathcal{C}$.

(i) *The pair $(\mathcal{X}, P)$ is said to verify the "glueing property" if any coherent family of elements*
$$(p_i \in P(X_i))_{i\in I}$$
lifts to a unique element $p \in P(X)$.

(ii) *The pair $(\mathcal{X}, P)$ is said to verify the "stable glueing property" if, for any presieve $\mathcal{X}'$ on a city X' of $\mathcal{C}$ which is a pull-back of $\mathcal{X}$ by an itinerary $X' \xrightarrow{x} X$, the pair $(\mathcal{X}', P)$ verifies the "glueing property".*

Grothendieck topologies, subtoposes and their duality

Start with $C =$ class of presieves $(X_i \xrightarrow{x_i} X)_{i \in I}$ of a category $\mathcal{C}$,

$S =$ class of presheaves on $\mathcal{C}$,

and $R \hookrightarrow C \times S$ defined by the "stable glueing property", inducing

$$(O_C, \subseteq) \underset{G_R}{\overset{F_R}{\rightleftarrows}} (O_S, \supseteq)\,.$$

Proposition (O. Caramello, L.L.). –

(i) *The fixed points of $G_R \circ F_R$ in O_C are exactly the subclasses $J \subseteq C$ which are "Grothendieck topologies" in the sense that*

  (1) *For any city X of $\mathcal{C}$, $(X \xrightarrow{\mathrm{id}_X} X)$ belongs to J.*

  (2) *For any element $\mathcal{X}$ of J, any pull-back of $\mathcal{X}$ by any itinerary $X' \xrightarrow{x} X$ also belongs to J.*

  (3) *If $(X_i \xrightarrow{x_i} X)_{i \in I}$ belongs to J, as well as $(X_{i,j} \xrightarrow{x_{i,j}} X_i)_{j \in I_i}$, $\forall\, i \in I$, then $(X_{i,j} \xrightarrow{x_i \circ x_{i,j}} X)_{\substack{i \in I \\ j \in I_i}}$ belongs to J.*

(ii) *The fixed points of $F_R \circ G_R$ in O_S are exactly the subclasses $T \subseteq S = \{$presheaves on $\mathcal{C}\}$ whose associated full subcategory $\mathcal{E}_T \hookrightarrow \widehat{\mathcal{C}}$ is a "subtopos" in the sense that the embedding functor*

$$j_* : \mathcal{E}_T \longhookrightarrow \widehat{\mathcal{C}}$$

has a left adjoint

$$j^* : \widehat{\mathcal{C}} \longrightarrow \mathcal{E}_T$$

which respects finite limits.

The closure property

Definition. –
A subpresheaf $Q \subseteq P$ of a presheaf P on a category $\mathcal{C}$
is a family of subsets

$$Q(X) \subseteq P(X) \quad \textit{indexed by cities } X \textit{ of } \mathcal{C}$$

such that, for any itinerary $X' \xrightarrow{x} X$ of $\mathcal{C}$, the map

$$P(x) : P(X) \longrightarrow P(X')$$

restricts to

$$Q(X) \longrightarrow Q(X') \,.$$

Definition. –
Let $\mathcal{X} = (X_i \xrightarrow{x_i} X)_{i \in I}$ be a presieve on a city X of $\mathcal{C}$
and $Q \subseteq P$ be a subpresheaf of a presheaf P on $\mathcal{C}$.

(i) *The pair $(\mathcal{X}, Q \subseteq P)$ is said to verify the "closure property"*
if any elements

$$p \in P(X) \quad \textit{such that} \quad P(x_i)(p) \in Q(X_i) \subseteq P(X_i) \,, \; \forall\, i \in I,$$

is necessarily in the subset $\quad Q(X) \subseteq P(X)$.

(ii) *The pair $(\mathcal{X}, Q \subseteq P)$ is said to verify the "stable closure property"*
if, for any presieve $\mathcal{X}'$ on a city X' of $\mathcal{C}$
which is a pull-back of $\mathcal{X}$ by an itinerary $X' \xrightarrow{x} X$,
the pair $(\mathcal{X}', Q \subseteq P)$ verifies the "closure property".

The duality of Grothendieck topologies and closure operators

Start with C = class of presieves $(X_i \xrightarrow{x_i} X)_{i \in I}$ of a category $\mathcal{C}$,

S = class of subpresheaves $Q \subseteq P$ of presheaves P on $\mathcal{C}$,

and $R \hookrightarrow C \times S$ defined by the "stable closure property", inducing

$$(O_C, \subseteq) \underset{G_R}{\overset{F_R}{\rightleftarrows}} (O_S, \supseteq) .$$

Proposition (O. Caramello, L.L.). –

(i) *The fixed points of $G_R \circ F_R$ in O_C are exactly the subclasses $J \subseteq C$ which are Grothendieck topologies.*

(ii) *The fixed points of $F_R \circ G_R$ in O_S are exactly the subclasses $T \subseteq S$ which verify:*

(1) *All full subpresheaves $P = P$ are in T.*

(2) *For any family of subpresheaves $Q_i \subseteq P$, $i \in I$, which are in T, their intersection*

$$\bigcap_{i \in I} Q_i \subseteq P \qquad \text{is in } T.$$

In particular, any subpresheaf $Q \subseteq P$ generates a smallest subpresheaf $\overline{Q} \subseteq P$ containing Q which is in T.

(3) *For any morphism of presheaves $P' \xrightarrow{u} P$, the pull-back map*

$$u^{-1} : \{\text{subpresheaves of } P\} \longrightarrow \{\text{subpresheaves of } P'\}$$

respects the closure operators $Q \mapsto \overline{Q}$ in the sense that, for $Q \subseteq P$,

$$u^{-1}(\overline{Q}) = \overline{u^{-1}(Q)}.$$