From topology to logic and provability through Grothendieck topos theory

Laurent LAFFORGUE

(Lagrange Centre, Huawei Technologies France)

Peking University, Beijing, Thursday December 11^{th}, 2025

From metric spaces to topological spaces

- The general notion of "metric space"
  ($X =$ set, $d =$ distance map on pairs of points of X
  respecting $\quad d(x_1, x_2) + d(x_2, x_3) \geq d(x_1, x_3)$)
  has been defined by Fréchet in 1906.
- The general notion of "topological space"
  ($X =$ set, $\mathcal{O}_X =$ family of "open" subsets of X
  stable under arbitrary union and finite intersection)
  has been defined by Hausdorff in 1914.

Observations:

- Any metric space (X, d) defines a topological space $(X, \mathcal{O}_X)$
  by deciding that a subset $U \subseteq X$ is "open" if and only if

$$\forall\, x \in U, \exists\, \varepsilon > 0, \forall\, x' \in X, d(x', x) < \varepsilon \Rightarrow x' \in U.$$

- A map between metric spaces

$$f : (X', d') \longrightarrow (X, d)$$

  is continuous in the metric sense
  if and only if it is continuous in the topological sense.
- Different metrics d on a set X may define the same topology $\mathcal{O}_X$.
- Many topologies $\mathcal{O}_X$ on sets X are not defined by metrics d.

Grothendieck's reinterpretation of the notion of topology

For Grothendieck, the structure of topological space

$$(X, \mathcal{O}_X)$$

consists in two parts:

- on the one hand, a set $\mathcal{O}_X$ partially ordered by inclusion

$$U' \subseteq U$$

- on the other hand, a notion of covering of elements U of $\mathcal{O}_X$

  by smaller elements

$$U_i \subseteq U, \quad i \in I,$$

  defined by the condition $\quad \bigcup_{i \in I} U_i = U.$

Observations: Any partially ordered set

$$(\mathcal{O}, \leq)$$

can be seen as a category $\mathcal{C}$ such that

- the "objects" of $\mathcal{C}$ are the elements of $\mathcal{O}$,
- the morphisms $\quad U' \to U \quad$ of $\mathcal{C}$

  correspond to relations $\quad U' \leq U \quad$ in $\mathcal{O}$,
- the composition law of morphisms of $\mathcal{C}$

  correspond to the transitivity property of the relation $\leq$.

The generalized notion of topology on a category

Definition (Grothendieck). – *A topology J on a category $\mathcal{C}$ consists in a notion of "covering" of objects X of $\mathcal{C}$ by families of morphisms to X*

$$X_i \xrightarrow{x_i} X\,, \quad i \in I\,,$$

such that the following axioms are verified by J:

(0) *For any J-covering* $(X_i \xrightarrow{x_i} X)_{i \in I}$,

a family of morphisms $(X'_{i'} \xrightarrow{x'_{i'}} X)_{i' \in I'}$ *is a J-covering if any morphism*

$$X_i \xrightarrow{x_i} X\,, \quad i \in I$$

factorizes through at least one $X'_{i'} \xrightarrow{x'_{i'}} X$, $i' \in I'$, *as*

$$x_i : X_i \longrightarrow X'_{i'} \xrightarrow{x'_{i'}} X\,.$$

(1) *For any object X, $X \xrightarrow{\mathrm{id}} X$ is a J-covering.*

(2) *For any J-covering $(X_i \xrightarrow{x_i} X)_{i \in I}$ and any morphism $X' \xrightarrow{x} X$, there exists a J-covering $(X'_{i'} \xrightarrow{x'_{i'}} X')_{i' \in I'}$ whose elements $X'_{i'} \xrightarrow{x'_{i'}} X'$ are part of at least one commutative square*

$$\begin{tikzcd} X'_{i'} \arrow[r] \arrow[d, "x'_{i'}"'] & X_i \arrow[d, "x_i"] \\ X' \arrow[r, "x"] & X \end{tikzcd} \quad \textit{for some} \quad i \in I\,.$$

(3) *If $(X_i \xrightarrow{x_i} X)_{i \in I}$ is a J-covering and $(X_{i,j} \xrightarrow{x_{i,j}} X_i)_{j \in I_i}$ is a J-covering for any $i \in I$, then $(X_{i,j} \xrightarrow{x_{i,j} \circ x_i} X)_{i \in I, j \in I_i}$ is a J-covering.*

Sheaves as continuous structure-reflecting set-valued functions

Definition. – *A "presheaf" on a category $\mathcal{C}$ is a structure-reflecting set-valued function on $\mathcal{C}$*

$$P : \begin{cases} \textit{objects } X \longmapsto P(X) = \textit{sets}, \\ \textit{morphisms } (X' \xrightarrow{x} X) \longmapsto (P(X) \xrightarrow{P(x)} P(X')) = \textit{"restriction" maps}, \end{cases}$$

such that

$$\begin{cases} \bullet \quad \textit{P transforms compositions} \quad X'' \xrightarrow{x'} X' \xrightarrow{x} X \\ \quad\ \ \textit{into compositions} \quad P(X) \xrightarrow{P(x)} P(X') \xrightarrow{P(x')} P(X''), \\ \bullet \quad \textit{P transforms identities } X \xrightarrow{\mathrm{id}} X \textit{ into identities } P(X) \xrightarrow{\mathrm{id}} P(X). \end{cases}$$

Definition. – *A presheaf on a category $\mathcal{C}$* $\qquad P : \mathcal{C} \longrightarrow \mathrm{Set} \qquad$ *is a "J-sheaf" for a topology J on $\mathcal{C}$ if it is "J-continuous" in the sense that, for any J-covering*

$$(X_i \xrightarrow{x_i} X)_{i \in I}\,,$$

elements $p \in P(X)$ correspond one-to-one to families of elements

$$p_i \in P(X_i)\,, \quad i \in I\,,$$

which are "compatible" in the sense that any commutative square

$$\begin{tikzcd} U \arrow[r, "u_2"] \arrow[d, "u_1"'] & X_{i_2} \arrow[d, "x_{i_2}"] \\ X_{i_1} \arrow[r, "x_{i_1}"] & X \end{tikzcd} \qquad \textit{yields an equality} \quad P(u_1)(p_{i_1}) = P(u_1)(p_{i_2})\,.$$

Categories of presheaves or sheaves

Definition. –

(i) *For any category $\mathcal{C}$, presheaves on $\mathcal{C}$ make up a category whose morphisms* $(\mathcal{C}^{\mathrm{op}} \xrightarrow{P} \mathrm{Set}) \xrightarrow{a} (\mathcal{C}^{\mathrm{op}} \xrightarrow{Q} \mathrm{Set})$ *are families of maps*

$$a_X : P(X) \longrightarrow Q(X)\,, \quad X = \textit{object of } \mathcal{C},$$

such that any morphism $X' \xrightarrow{x} X$ of $\mathcal{C}$ induces a commutative square:

$$\begin{tikzcd} P(X) \arrow[r, "P(x)"] \arrow[d, "a_X"'] & P(X') \arrow[d, "a_{X'}"] \\ Q(X) \arrow[r, "Q(x)"] & Q(X') \end{tikzcd}$$

(ii) *For any topology J on a category $\mathcal{C}$, J-sheaves on $\mathcal{C}$ make up a full subcategory*

$$\widehat{\mathcal{C}}_J \hookrightarrow \widehat{\mathcal{C}}$$

whose morphisms are morphisms of presheaves.

Definition. –

A topos is a category which is equivalent to some

$$\widehat{\mathcal{C}} \ \textit{ or more generally } \ \widehat{\mathcal{C}}_J$$

for some small category $\mathcal{C}$.

A device for making functions continuous

Proposition. –

Consider a topology J on a (essentially) small category $\mathcal{C}$.

(i) *The embedding functor*

$$j_* : \widehat{\mathcal{C}}_J \hookrightarrow \widehat{\mathcal{C}}$$

admits a left-adjoint, called the J-sheafification functor

$$j^* : \widehat{\mathcal{C}} \longrightarrow \widehat{\mathcal{C}}_J \,.$$

(ii) *The natural transform $j^* \circ j_* \to \mathrm{Id}_{\widehat{\mathcal{C}}_J}$ is an isomorphism, meaning the J-sheafification of any J-sheaf is itself.*

(iii) *Sheafification by j^* respects arbitrary colimits and finite limits.*

Remark:

If presheaves on $\mathcal{C}$ are seen as set-valued (structure-reflecting) functions
and J-sheaves as continuous set-valued functions,
the functor

$$j^* : \widehat{\mathcal{C}} \longrightarrow \widehat{\mathcal{C}}_J$$

can be seen as a device
which transforms any (set-valued structure-reflecting) function
into a J-continuous function.

Toposes as analogs of Banach spaces

Lemma. – *Let $\mathcal{C}$ be a (essentially) small category.*

(i) *(Yoneda) The functor*

$$\begin{array}{rcl} y : \mathcal{C} & \longrightarrow & \widehat{\mathcal{C}} \\ X & \longmapsto & \mathrm{Hom}(\bullet, X) = \left\{ \begin{array}{ccc} X' & \longmapsto & \mathrm{Hom}(X', X) \\ \| & & \| \\ \textit{object of } \mathcal{C} & & \textit{set of morphisms } X' \to X \end{array} \right. \end{array}$$

is fully faithful, i.e. identifies $\mathcal{C}$ to a full subcategory of $\widehat{\mathcal{C}}$.

(ii) *Any object P of $\widehat{\mathcal{C}}$ can be represented as a colimit* $\quad P = \varinjlim\limits_{(X,x)} y(X)$

indexed by the category of pairs $(X \in \mathrm{Ob}(\mathcal{C}), x \in P(X))$.

Corollary. – *Let J be a topology on a (essentially) small category $\mathcal{C}$.*

(i) *The categories $\mathcal{C}$ and $\widehat{\mathcal{C}}_J$ are related by the functor* $\quad \ell : j^* \circ y : \mathcal{C} \xhookrightarrow{y} \widehat{\mathcal{C}} \xrightarrow{j^*} \widehat{\mathcal{C}}_J\,.$

(ii) *Any J-sheaf F on $\mathcal{C}$ can be represented as a colimit* $\quad F = \varinjlim\limits_{(X,x)} \ell(X).$

Comment: This means that $\widehat{\mathcal{C}}$ or $\widehat{\mathcal{C}}_J$ are deduced from $\mathcal{C}$ by completion
just as Banach spaces are deduced by completion
from spaces of functions endowed with a chosen topology.

Remark: Grothendieck worked first in functional analysis.

A topossic analog of Gelfand duality

Theorem. –

(i) *Any topological space X defines a topos*

$$\mathcal{E}_X = \widehat{(\mathcal{O}_X)}_{J_X}.$$

(ii) *Any continuous map between topological spaces*

$$X' \xrightarrow{f} X$$

defines a pair of adjoint functors

$$(\mathcal{E}_X \xrightarrow{f^*} \mathcal{E}_{X'}, \mathcal{E}_{X'} \xrightarrow{f_*} \mathcal{E}_X)$$

whose pull-back component

$$f^* : \mathcal{E}_X \longrightarrow \mathcal{E}_{X'}$$

respects arbitrary colimits and finite limits.

(iii) *The correspondence from continuous maps to such pairs of adjoint functors*

$$(X' \to X) \longmapsto (\mathcal{E}_X \to \mathcal{E}_{X'}, \mathcal{E}_{X'} \to \mathcal{E}_X)$$

is one-to-one if the topological space X is "sober".

(iv) *In particular, a topological space X can be reconstructed from the associated topos* $\mathcal{E}_X$ *if X is sober.*

Remark:

If $\mathcal{E}_X$ and $\mathcal{E}_{X'}$ are seen as spaces of continuous set-valued functions on X and X', pull-back components $\mathcal{E}_X \to \mathcal{E}_{X'}$ can be seen as operators of change of parameters.

Grothendieck's metamorphosis of topology

For Grothendieck, toposes are the most general math objects
to which notions, intuitions and constructions of topological nature
still apply.

Definition. –
Generalizing the notion of continuous map between topological spaces,
a morphism of toposes

$$f : \mathcal{E}' \longrightarrow \mathcal{E}$$

is defined as a pair of adjoint functors

$$(\mathcal{E} \xrightarrow{f^*} \mathcal{E}', \mathcal{E}' \xrightarrow{f_*} \mathcal{E})$$

whose pull-back component f^* respects finite limits (in addition to colimits).

Remarks:

(i) Morphisms $\mathcal{E}' \to \mathcal{E}$
can also be called mobile points of $\mathcal{E}$ parametrized by $\mathcal{E}'$.

(ii) In particular, fixed points of a topos $\mathcal{E}$ are morphisms

$$\mathcal{E}_{\{\bullet\}} = \mathrm{Set} \longrightarrow \mathcal{E}.$$

(iii) In general, non-trivial toposes may have only mobile points,
without any fixed point.

The duality of toposes and their multiple geometric presentations

By definition, any topos $\mathcal{E}$ can be presented as a topos of sheaves

$$\mathcal{E} \xrightarrow{\sim} \widehat{\mathcal{C}}_J$$

on some (essentially) small category $\mathcal{C}$ endowed with a topology J.
There are infinitely many such presentations:

Proposition ("Grothendieck's comparison lemma"). – *Let J be a topology on $\mathcal{C}$.*
Consider a full subcategory $\mathcal{C}' \hookrightarrow \mathcal{C}$
which is "dense" in the sense that any object X of $\mathcal{C}$ admits a J-covering

$$(X_i \xrightarrow{x_i} X)_{i \in I} \quad \text{by objects} \quad X_i \text{ of } \mathcal{C}'.$$

If J' is the topology on $\mathcal{C}'$ induced by J, restriction from $\mathcal{C}$ to $\mathcal{C}'$ induces an equivalence $\widehat{\mathcal{C}}_J \xrightarrow{\sim} \widehat{\mathcal{C}'}_{J'}$.

Theorem. –
Any topos $\mathcal{E}$ identifies with the category of sheaves on $\mathcal{E}$
for the "canonical topology" (for which coverings are globally epimorphic families).

Corollary. –
Any full subcategory of a topos $\mathcal{E}$

$$\mathcal{C} \longhookrightarrow \mathcal{E} \quad \text{which is dense}$$

defines a presentation of $\mathcal{E}$

$$\mathcal{E} \xrightarrow{\sim} \widehat{\mathcal{C}}_J$$

by $\mathcal{C}$ endowed with the canonical topology of $\mathcal{E}$.

Presentations and approximations of points of toposes

Theorem (Grothendieck, Diaconescu). –
Consider a topos $\mathcal{E}$ presented as $\mathcal{E} \xrightarrow{\sim} \widehat{\mathcal{C}}_J$.
Then, the category of mobile points $\mathcal{E}' \to \mathcal{E}$ parametrized by a topos $\mathcal{E}'$
is equivalent to the category of functors

$$\rho : \mathcal{C} \longrightarrow \mathcal{E}'$$

which are

- *"flat",*
- *J-continuous in the sense that they transform J-covering families of $\mathcal{C}$ into globally epimorphic families of $\mathcal{E}'$.*

Remark: In the case where $\mathcal{C}$ has finite limits,
a functor $\mathcal{C} \xrightarrow{\rho} \mathcal{E}'$ is "flat" if and only if it respects finite limits.

Definition. –
An approximation of a point

$$\mathcal{E}' \longrightarrow \mathcal{E} \xrightarrow{\sim} \widehat{\mathcal{C}}_J$$

presented as a "flat" J-continuous functor

$$\rho : \mathcal{C} \longrightarrow \mathcal{E}'$$

can be defined as a restriction of ρ
to any finite subdiagram D of $\mathcal{C}$.

Remark: This definition generalizes the usual notion of approximation
of real numbers or real-valued continuous functions.

First-order (geometric) constructions in Set or in a topos

- Considering sets or more generally objects of a topos $\mathcal{E}$,
  and giving names to them (ex: ring R, module M, $\cdots$).
- Forming (finite) products of objects $E_1 \times \cdots \times E_n$ characterized by the one-to-one correspondence
  $$(E \xrightarrow{e} E_1 \times \cdots \times E_n) \longleftrightarrow (E \xrightarrow{e_1} E_1, \cdots, E \xrightarrow{e_n} E_n),$$
  in particular the terminal object $\{\bullet\}$ or $1_{\mathcal{E}}$.
- Considering maps or more generally morphisms of $\mathcal{E}$ $E_1 \times \cdots \times E_n \longrightarrow F$
  and giving names to them (ex: multiplication, addition, $\cdots$).
- Composing maps or morphisms.
- Forming equalizers $\mathrm{eq}\Big(E_1 \times \cdots \times E_n \underset{g}{\overset{f}{\rightrightarrows}} F\Big)$ characterized by
  $$\Big(E \longrightarrow \mathrm{eq}\Big(E_1 \times \cdots \times E_n \underset{g}{\overset{f}{\rightrightarrows}} F\Big)\Big) \longleftrightarrow (E \xrightarrow{e} E_1 \times \cdots \times E_n \text{ such that } f \circ e = g \circ e).$$
- Combining products and equalizers, forming limits $\varprojlim\limits_{D}$ of (finite) diagrams D.
- Considering subsets or more generally subobjects, as equivalence classes
  of monomorphisms $R \hookrightarrow E_1 \times \cdots \times E_n$,
  and giving names to them (ex: order relation, belonging relation of a point to a line, $\cdots$)
- Forming (finite) intersections or arbitrary unions of subobjects,
  including the full subobject and the initial subobject $\emptyset$ or $\emptyset_{\mathcal{E}}$.
- Forming images of morphisms $R \longrightarrow E_1 \times \cdots \times E_n$ as subobjects
  $$\varinjlim \,(R \times_{E_1 \times \cdots \times E_n} R \rightrightarrows R) \hookrightarrow E_1 \times \cdots \times E_n\,.$$

Syntax and semantics of geometric first-order theories

Definition. –

(i) *A first-order language* Σ *consists in*

$$\begin{cases} \bullet & \text{a list of names of objects,} \\ \bullet & \text{a list of names of morphisms,} \\ \bullet & \text{a list of names of relations.} \end{cases}$$

(ii) *A first-order geometric theory* $\mathbb{T}$ *in a language* Σ *consists in a list of axioms of the form*

$$\varphi(x_1^{A_1}, \cdots, x_n^{A_n}) \vdash \psi(x_1^{A_1}, \cdots, x_n^{A_n})$$

which are implications between two formulas φ, ψ *in*

$$\begin{cases} \bullet & \text{formal variables } x_1^{A_1}, \cdots, x_n^{A_n} \text{ associated to names of objects } A_1, \cdots, A_n, \text{ such that:} \\ \bullet & \varphi, \psi \text{ are geometric processes for constructing subobjects } M\varphi, M\psi \\ & \text{in the product of objects } MA_1 \times \cdots \times MA_n \text{ named } A_1, \cdots, A_n. \end{cases}$$

Definition. –

Let $\mathbb{T}$ *be a geometric first-order theory in a language* Σ.

(i) *A* Σ*-model* M *in a topos* $\mathcal{E}$ *consists in*
objects MA, *morphisms* $MA_1 \times \cdots \times MA_n \to MB$, *relations* $MR \hookrightarrow MA_1 \times \cdots \times MA_n$
named after the elements of the vocabulary Σ.

(ii) *A* Σ*-model* M *is a* $\mathbb{T}$*-model if, for any axiom* $\varphi \vdash \psi$,
the corresponding subobjects verify $M\varphi \subseteq M\psi$.

Parametrized models and changes of parameters

Proposition. –
Let $\mathbb{T}$ be a first-order geometric theory.

(i) *For any topos $\mathcal{E}$, $\mathbb{T}$-models in $\mathcal{E}$ form a category*

$$\mathbb{T}\text{-mod}\,(\mathcal{E})$$

which can be seen as the category of
continuous or mobile models of $\mathbb{T}$
parametrized by the topos $\mathcal{E}$
(seen as a generalized topological space).

(ii) *For any topos morphism $\mathcal{E}' \xrightarrow{f} \mathcal{E}$ consisting in*

$$\left(\mathcal{E} \xrightarrow{f^*} \mathcal{E}', \mathcal{E}' \xrightarrow{f_*} \mathcal{E}\right),$$

its pull-back component f^* defines a functor

$$f^* : \mathbb{T}\text{-mod}\,(\mathcal{E}) \longrightarrow \mathbb{T}\text{-mod}\,(\mathcal{E}')$$

which can be seen as
an operation of change of parameters
from $\mathcal{E}$-parametrized continuous models of $\mathbb{T}$
to $\mathcal{E}'$-parametrized continuous models of $\mathbb{T}$
induced by the topos morphism $f : \mathcal{E}' \to \mathcal{E}$
(seens as a generalized continuous map).

The topossic incarnation of the semantics of theories

Theorem. –
For any geometric first-order theory $\mathbb{T}$,

there exists a topos $\mathcal{E}_{\mathbb{T}}$ *(unique up to equivalence)*

endowed with a $\mathbb{T}$*-model* $U_{\mathbb{T}}$

which is universal in the sense that,

for any topos $\mathcal{E}$, *the functor*

$$\begin{array}{rcl} \{\textit{morphisms } \mathcal{E} \to \mathcal{E}_{\mathbb{T}}\} & \longrightarrow & \mathbb{T}\text{-mod}(\mathcal{E}) \,, \\ (f : \mathcal{E} \to \mathcal{E}_{\mathbb{T}}) & \longmapsto & f^* \, U_{\mathbb{T}} \end{array}$$

is an equivalence of categories.

Remark:
In other words, $\mathcal{E}$-parametrized continuous models of $\mathbb{T}$
interpret as $\mathcal{E}$-parametrized mobile points of $\mathcal{E}_{\mathbb{T}}$,
called the "classifying topos" of $\mathbb{T}$.

Theorem. –
For any topology J *on a (essentially) small category* $\mathcal{C}$,

the theory $\mathbb{T}_{\mathcal{C},J}$ *of "flat* J*-continuous functors" on* $\mathcal{C}$

is geometric first-order

and its classifying topos is

$$\mathcal{E}_{\mathbb{T}_{\mathcal{C},J}} \cong \widehat{\mathcal{C}}_J \,.$$

The notion of subtopos

Definition. –

(i) *A topos morphism* $j : \mathcal{E}' \to \mathcal{E}$*, consisting in a pair*

$$(\mathcal{E} \xrightarrow{j^*} \mathcal{E}', \mathcal{E}' \xrightarrow{j_*} \mathcal{E}),$$

is called an "embedding" if and only if its push-forward component

$$j_* : \mathcal{E}' \longrightarrow \mathcal{E}$$

is fully faithful.

(ii) *A subtopos of a topos* $\mathcal{E}$ *is an equivalence class of embeddings*

$$j : \mathcal{E}' \hookrightarrow \mathcal{E},$$

considered up to equivalences of toposes $\mathcal{E}'$.

Any subtopos has a canonical representative, which leads to the following equivalent definition:

Definition. –

A subtopos of a topos $\mathcal{E}$ *is a full subcategory*

$$\mathcal{E}' \hookrightarrow \mathcal{E}$$

such that:

(1) *The embedding functor* $j_* : \mathcal{E}' \hookrightarrow \mathcal{E}$ *has a left adjoint* $\mathcal{E}' \xrightarrow{j^*} \mathcal{E}$.

(2) *This adjoint* $j^* : \mathcal{E} \to \mathcal{E}'$ *respects finite limits in addition to colimits.*

(3) *An object* E *of* $\mathcal{E}$ *is in* $\mathcal{E}'$ *if and only if the canonical morphism*

$$E \longrightarrow j_* \circ j^* E \qquad \text{is an isomorphism.}$$

Topological and logical expressions of subtoposes

Theorem (Grothendieck, SGA 4). –

For any presentation of a topos $\mathcal{E}$ as a topos of sheaves

$$\mathcal{E} \xrightarrow{\sim} \widehat{\mathcal{C}}_J \,,$$

subtoposes $\quad \mathcal{E}' \hookrightarrow \mathcal{E} \quad$ *correspond one-to-one to*

topologies $\quad J' \supseteq J \quad$ *on* $\mathcal{C}$.

Theorem (O. Caramello, see her book "Theories, Sites, Toposes"). –

For any geometric first-order theory $\mathbb{T}$ in a language Σ,

subtoposes

$$\mathcal{E}' \longhookrightarrow \mathcal{E}_{\mathbb{T}}$$

correspond one-to-one to

equivalence classes (for the relation of provable equivalence)

of "quotient" theories $\mathbb{T}'$ of $\mathbb{T}$,

i.e. theories in the same language Σ

derived from $\mathbb{T}$ by adding a family of extra axioms.

Inner and outer operations on subtoposes

Theorem. –

(i) *For any topos $\mathcal{E}$, subtoposes on $\mathcal{E}$ make up a set $\mathrm{Sub}(\mathcal{E})$ partially ordered by inclusion, with arbitrary unions and intersections.*

(ii) *Any pair of subtoposes* $\mathcal{E}_1 \hookrightarrow \mathcal{E}, \mathcal{E}_2 \hookrightarrow \mathcal{E}$
define a difference subtopos $\mathcal{E}_1 \backslash \mathcal{E}_2 \hookrightarrow \mathcal{E}$
characterized by $\mathcal{E}' \supseteq \mathcal{E}_1 \backslash \mathcal{E}_2 \Leftrightarrow \mathcal{E}' \cup \mathcal{E}_2 \supseteq \mathcal{E}_1$.

Remarks:

- (ii) implies that the operation $\mathcal{E}' \mapsto \mathcal{E}' \cup \mathcal{E}_2$ respects arbitrary intersections.
- As a consequence, the operation $\mathcal{E}' \mapsto \mathcal{E}' \cap \mathcal{E}_1$ respects finite unions.

Theorem. –

(i) *Any topos morphism* $f : \mathcal{E}' \to \mathcal{E}$
defines a push-forward transform

$$\begin{array}{rcl} \exists_f = f_* : \mathrm{Sub}(\mathcal{E}') & \longrightarrow & \mathrm{Sub}(\mathcal{E})\,, \\ (\mathcal{E}'_1 \hookrightarrow \mathcal{E}') & \longmapsto & \mathrm{Im}(\mathcal{E}'_1 \hookrightarrow \mathcal{E}' \to \mathcal{E}) \hookrightarrow \mathcal{E}\,. \end{array}$$

(ii) *This push-forward transform f_* has a left adjoint*

$$f^* = f^{-1} : \mathrm{Sub}(\mathcal{E}) \longrightarrow \mathrm{Sub}(\mathcal{E}')\,,$$

called the pull-back transform, which respects finite intersections (O.C., L.L.) in addition to arbitrary unions.

(iii) *(O.C., L.L.) If f is "locally connected", f^* has a left adjoint*

$$\forall_f = f_! : \mathrm{Sub}(\mathcal{E}') \longrightarrow \mathrm{Sub}(\mathcal{E})\,.$$

Equivalence of first-order geometric provability and topology generation

Topological and logical presentations. –

(i) *For any presentation* $\mathcal{E} \xrightarrow{\sim} \widehat{\mathcal{C}}_J$, *subtoposes* $\mathcal{E}' \hookrightarrow \mathcal{E}$ *correspond to topologies* $J' \supseteq J$ *on* $\mathcal{C}$, *concretely given by families of coverings*

$$(X_i \xrightarrow{x_i} X)_{i \in I} \quad \text{which belong to } J' \text{ and generate } J' \text{ on } J.$$

(ii) *For any presentation* $\mathcal{E} \xrightarrow{\sim} \mathcal{E}_{\mathbb{T}}$, *subtoposes* $\mathcal{E}' \hookrightarrow \mathcal{E}$ *correspond to quotient theories* $\mathbb{T}'$ *of* $\mathbb{T}$, *concretely given by families of extra axioms in the same language*

$$\varphi \vdash \psi \quad \text{which belong to } \mathbb{T}' \text{ and generate } \mathbb{T}' \text{ over } \mathbb{T}.$$

(iii) *Any equivalence* $\widehat{\mathcal{C}}_J \cong \mathcal{E}_{\mathbb{T}}$ *defines a double constructive translation process*

- *from covering families to axioms,*
- *from axioms to covering families,*
- *between logical deductions, making up proofs, and applications of the generating rules of Grothendieck topologies.*

Remarks:

- Point-based mathematics, in terms of elements of spaces transformed by functions, is naturally formulated in terms of linear sequences of symbols.
- Subtoposes and operations on subtoposes are not linear.
  They are more space-like.
  They are made more or less precise by adding or subtracting components.

Known facts. –

- *Any result in higher order mathematics can be constructively reduced to a result in first-order mathematics.*
- *Any first-order theory* $\mathbb{T}$ *can be constructively transformed in a geometric first-order theory* $\mathbb{T}_g$ *without changing its set-based models.*
- *A property formulated in the language* Σ *of a geometric theory* $\mathbb{T}$ *is provable in* $\mathbb{T}$ *if and only if it is verified by the universal model* $U_{\mathbb{T}}$ *of* $\mathbb{T}$ *in* $\mathcal{E}_{\mathbb{T}}$.
- *Any geometric first-order theory* $\mathbb{T}$ *of language* Σ *can be constructively transformed into a "relational" geometric first-order theory* $\mathbb{T}_r$ *(whose language* Σ_r *has only relation names, without morphism names) such that* $(\mathcal{E}_{\mathbb{T}}, U_{\mathbb{T}}) \cong (\mathcal{E}_{\mathbb{T}_r}, U_{\mathbb{T}_r})$.

Proposition (spelled out in detail in the book of O.C., L.L. on subtoposes). –
If Σ_r *is a relational language, its classifying topos has the form*

$$\mathcal{E}_{\Sigma_r} \cong \widehat{\mathcal{C}}$$

for a small category $\mathcal{C}$ *which is "locally computable" in the sense that*

- $\mathcal{C}$ *has arbitrary finite limits,*
- *objects* X *of* $\mathcal{C}$ *are parametrized by explicit combinatorial data,*
- *morphisms* $X \to Y$ *make up finite sets* $\mathrm{Hom}(X, Y)$ *which are computable.*