The interplay of semantics and topology through Grothendieck topos theory

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Structural AI Workshop

Shanghai Qi Zhi Institute, June 13-16, 2025

A mathematical notion of semantics

Starting remark: the notion of semantics of natural languages

- A natural language consists in
  - — a vocabulary (names, verbs, adjectives, prepositions, $\cdots$),
  - — grammar rules.
- The semantics of a natural language consists in
  - — primarily, the beings to which this language applies,
  - — ultimately, the meanings of all possible sentences and texts expressed in this language which make sense.

From natural languages to logic and mathematics:

- A formal language consists in
  - — a vocabulary (ex: “group”, “multiplication law”, “unit element”, “inverse”),
  - — a list of axioms (ex: “associativity”, “neutrality”, “inverse characterization”).
- The semantics of a formal language consists in
  - — firstly, the structures arising from mathematical practice to which such a formal language applies (ex: groups of symmetries),
  - — ultimately, all math structures to which the given formal language applies, together with their mutual relations (ex: all set-based groups, related by all group homomorphisms).

The mathematical definition of the semantics of a formal language

Definition (Tarski). –
Let $\mathbb{T}$ be a formal language (ex: group theory).
Then the (set-based) semantics of $\mathbb{T}$ consists in

$$\begin{cases} - & \textit{sets endowed with a structure of type } \mathbb{T} \textit{ (ex: a group structure),} \\ - & \textit{maps between such sets which respect their structures.} \end{cases}$$

Remark:
Sets endowed with structures of type $\mathbb{T}$ and compatible maps make up

$$\mathbb{T}\text{-mod (Set)} = \text{“category” of set-based “models” of } \mathbb{T}.$$

Generalization based on geometric practice:

- For “good” parameter spaces $\mathcal{E}$, there should also exist

$$\mathbb{T}\text{-mod }(\mathcal{E}) = \text{“category” of } \mathcal{E}\text{-valued models of } \mathbb{T},$$

i.e. of “continuous families” of models of $\mathbb{T}$ parametrized by $\mathcal{E}$
(ex: group bundles on a topological space).

- For good “change of parameters” maps between such parameter spaces

$$e : \mathcal{E}' \longrightarrow \mathcal{E}\,,$$

there should exist functors of change of parameters

$$e^* : \mathbb{T}\text{-mod}(\mathcal{E}) \longrightarrow \mathbb{T}\text{-mod}(\mathcal{E}')\,.$$

Grothendieck toposes as good parameter spaces

Expectations for "good" parameter spaces $\mathcal{E}$:

• They should generalize the category Set.

• They should be categories $\mathcal{E}$ with enough good properties so that $\mathbb{T}\text{-mod}\,(\mathcal{E}) = \{$models of $\mathbb{T}$ parametrized by $\mathbb{T}\}$ are well-defined as categories for all formal languages $\mathbb{T}$.

• The usual vocabulary of geometry ("points", "subspace", $\cdots$) should apply to them.

• They should be related by "geometric transforms"

$$e : \mathcal{E}' \longrightarrow \mathcal{E}\,,$$

inducing "change of parameters" functors

$$e^* : \mathbb{T}\text{-mod}(\mathcal{E}) \longrightarrow \mathbb{T}\text{-mod}(\mathcal{E}')\,.$$

Realization of these expectations:

The notions of Grothendieck topos $\mathcal{E}$

and topos map $\quad e : \mathcal{E}' \longrightarrow \mathcal{E}$

fulfill all these expectations.

Toposes as generalizations of topological spaces

Theorem (Grothendieck). –
(i) Any topological space X defines a topos

$$\mathcal{E}_X$$

so that, in particular,

$$\mathcal{E}_X = \mathrm{Set} \quad \text{if} \quad X = \{\bullet\}.$$

(ii) Any continuous map between topological spaces

$$s : S \longrightarrow X$$

defines a "topos transform"

$$\mathcal{E}_S \longrightarrow \mathcal{E}_X$$

(and conversely, if X is "sober", any topos transform

$$\mathcal{E}_S \longrightarrow \mathcal{E}_X$$

is induced by a unique continuous map

$$s : S \longrightarrow X).$$

Remarks:

- The notion of topos is much wider than the notion of topological space.
- A "point" of a topos $\mathcal{E}$ is defined as a topos transform

$$\mathrm{Set} \longrightarrow \mathcal{E}$$

(corresponding to $\{\bullet\} \to X$ if $\mathcal{E} = \mathcal{E}_X$).

From models to points: the topological incarnation of semantics

Definition. – *For any toposes $\mathcal{E}, \mathcal{E}'$, topos transforms*

$$\mathcal{E}' \longrightarrow \mathcal{E}\,,$$

are also called $\mathcal{E}'$-valued (or $\mathcal{E}'$-parametrized) mobile points of $\mathcal{E}$.

Remark: $\mathcal{E}'$-valued points of $\mathcal{E}$ make up a category $\mathrm{Geom}(\mathcal{E}', \mathcal{E})$.

Theorem (proved in the mid 1970's by categorical logicians generalizing ideas of Grothendieck and his student Monique Hakim). –
For any formal language $\mathbb{T}$ (belonging to first-order geometric logic),
there exists an associated topos

$$\mathcal{E}_{\mathbb{T}} = \text{“classifying topos” of } \mathbb{T}$$

such that:
(i) for any topos $\mathcal{E}$, the category of $\mathcal{E}$-valued models of $\mathbb{T}$

$$\mathbb{T}\text{-mod}\,(\mathcal{E})$$

is equivalent to the category of $\mathcal{E}$-parametrized points of $\mathcal{E}_{\mathbb{T}}$

$$\mathrm{Geom}\,(\mathcal{E}, \mathcal{E}_{\mathbb{T}}),$$

(ii) for any topos transform $\quad e : \mathcal{E}' \to \mathcal{E}$
the “change of parameter” functor $\quad e^* : \mathbb{T}\text{-mod}(\mathcal{E}) \to \mathbb{T}\text{-mod}(\mathcal{E}')$
corresponds to composition $\quad (\mathcal{E} \xrightarrow{m} \mathcal{E}_{\mathbb{T}}) \longmapsto (\mathcal{E}' \xrightarrow{e} \mathcal{E} \xrightarrow{m} \mathcal{E}_{\mathbb{T}})$.

Sketching of toposes by “sites”

Fact. – *Any Grothendieck topos can be presented concretely by “sites” $(\mathcal{C}, J)$ consisting in*

- *a small category $\mathcal{C}$,*
- *a Grothendieck topology J on $\mathcal{C}$ consisting in “covering families” of objects X of $\mathcal{C}$*

$$\left(X_i \xrightarrow{x_i} X\right)_{i \in I}$$

verifying stability under

$$\begin{cases} - & \text{induction through arrows } X' \xrightarrow{x} X \text{ of } \mathcal{C}, \\ - & \text{composition of coverings.} \end{cases}$$

Remarks:

- Any topos $\mathcal{E}$ has infinitely many sketches $(\mathcal{C}, J)$.
- A topos $\mathcal{E}$ is recovered from a sketch $(\mathcal{C}, J)$ by a uniform process
$$(\mathcal{C}, J) \longmapsto \widehat{\mathcal{C}}_J = \{\text{category of } J\text{-sheaves on } \mathcal{C}\} \cong \mathcal{E}$$
which can be considered as a completion process,
meaning in particular that there is a canonical functor
$$\mathcal{C} \longrightarrow \widehat{\mathcal{C}}_J \cong \mathcal{E}\,.$$

The concrete description of points

Theorem (Grothendieck, Diaconescu). –
Consider

$$\begin{cases} \bullet & \text{a small category } \mathcal{C}, \\ \bullet & \text{a topology } J \text{ on } \mathcal{C}, \end{cases}$$

and the associated topos $\widehat{\mathcal{C}}_J$.
Then the category

$$\mathrm{Geom}\,(\mathcal{E}, \widehat{\mathcal{C}}_J) = \{\mathcal{E} \to \widehat{\mathcal{C}}_J\}$$

of points of $\widehat{\mathcal{C}}_J$ *valued in a topos* $\mathcal{E}$,
is equivalent to the category of functors

$$\begin{array}{rcl} \rho : \mathcal{C} & \longrightarrow & \mathcal{E}\,, \\ X & \longmapsto & \rho(X)\,, \\ (X' \xrightarrow{x} X) & \longmapsto & (\rho(x) : \rho(X') \to \rho(X)) \end{array}$$

which are

$$\begin{cases} \bullet & \text{“flat”}, \\ \bullet & \text{“}J\text{-continuous”}. \end{cases}$$

The particular case of real numbers

Lemma. – *Let*
$Q =$ *any* <u>*subset of* $\mathbb{Q}$</u> *which is* <u>*dense*</u>*,*
$\mathcal{C} =$ *category of* <u>*intervals*</u> $]m, M[$*,* $m < M$*,* $m, M \in Q$*,*
$J =$ <u>*topology*</u> *of* <u>*coverings of intervals*</u>*.*
Then the topos $\widehat{\mathcal{C}}_J$
identifies with the <u>*topos defined by* $\mathbb{R}$</u>*.*

Application of Grothendieck-Diaconescu's equivalence:

A <u>real number</u> consists in a <u>compatible family</u>

$$\begin{array}{c}]m, M[\\ \| \\ \text{object of } \mathcal{C} \end{array} \longmapsto \text{subset of } \{\bullet\} = \begin{cases} \{\bullet\} = \text{“yes”}, \\ \text{or } \emptyset = \text{“no”}. \end{cases}$$

What this teaches us:

- In general, the <u>points of a topos</u> sketched by a site $(\mathcal{C}, J)$ are <u>just as abstract</u> as <u>real numbers</u>.
- The theory of <u>approximation of real numbers</u> (and numerical functions) <u>generalizes to arbitrary toposes</u>.

The notion of subtopos and its concrete expressions

Fact. – *There is a well-defined notion of "subtopos" of a topos* $\mathcal{E}$

$$\mathcal{E}' \hookrightarrow \mathcal{E}$$

such that

- *subtoposes of a topos* $\mathcal{E}$ *make up a set,*
- *this set is partially ordered by inclusion,*
- *families of subtoposes of* $\mathcal{E}$ *have a well-defined union or intersection.*

Theorem. –
(i) (Grothendieck) If $\mathcal{E} \cong \widehat{\mathcal{C}}_J$,
subtoposes $\mathcal{E}' \hookrightarrow \mathcal{E}$
correspond to topologies
J' *on* $\mathcal{C}$ *which are finer than* J.
(ii) (O. Caramello) If $\mathcal{E} \cong \mathcal{E}_{\mathbb{T}}$
is the "classifying topos" of a (first-order geometric) theory $\mathbb{T}$,
subtoposes $\mathcal{E}' \hookrightarrow \mathcal{E}$
correspond to "quotient theories" $\mathbb{T}'$ *of* $\mathbb{T}$
(i.e. theories with the same vocabulary and more axioms)
up to provable equivalence.

Application to provability problems

Consequence. –
Consider a formal language Σ.
Then it is possible to construct explicit fully computable categories

$$\mathcal{C}_\Sigma$$

and to write programs for

$$\begin{cases} \text{translating statements } S \text{ formulated in the language } \Sigma \\ \text{into coverings families } \mathcal{X}_S = (X_i \xrightarrow{x_i} X_S)_{i \in I_S} \text{ in } \mathcal{C}_\Sigma \end{cases}$$

so that

$$\begin{cases} \text{a statement } S \text{ is provable} \\ \text{from a family of axioms } S_k,\ k \in K, \end{cases}$$

if and only if

$$\begin{cases} \text{the covering family } \mathcal{X}_S \\ \text{belongs to the smallest topology } J \text{ on } \mathcal{C}_\Sigma, \\ \text{which contains the covering families } \mathcal{X}_{S_k},\ k \in K. \end{cases}$$

Remark:
This is the principle for new systems of Automatic Theorem Proving we are developping at the Lagrange Center.

The stuff toposes are made of

Definition. –
(i) For any small category $\mathcal{C}$ *(without topology),*
the associated topos $\widehat{\mathcal{C}}$
is the category of set-valued "presheaves" on $\mathcal{C}$

$$\begin{array}{rcl} P : \mathcal{C}^{\mathrm{op}} & \longrightarrow & \mathrm{Set}, \\ X & \longmapsto & P(X) \ (= \mathrm{Set}), \\ (X' \xrightarrow{x} X) & \longmapsto & (P(X) \xrightarrow{P(x)} P(X')) \ (= \textit{map in the reverse direction}). \end{array}$$

(ii) For any small category $\mathcal{C}$ *endowed with a topology* J*,*
the associated topos is the full subcategory

$$\widehat{\mathcal{C}}_J \hookrightarrow \widehat{\mathcal{C}}$$

of "J*-sheaves", i.e. of presheaves on* $\mathcal{C}$

$$P : \mathcal{C}^{op} \longrightarrow \mathrm{Set}$$

which are "continuous" with respect to the topology J *in some sense.*

Toposes as meaningful analogs of function spaces

Reminder. –
Functional Analysis consists in studying
spaces of numerical functions $C \to \mathbb{R}$
on some domains C.

Remarks:

- Spaces of numerical functions are amenable to rich computations (addition, multiplication, differentiation, integration, approximation).
- Their main drawback is that
  numbers and numerical functions are meaningless.

Geometry as meaning in toposes. –

- *A presheaf on a small category $\mathcal{C}$ can be interpreted as a function which is set-valued and geometrically structured.*
- *If it is a J-sheaf for a topology J on $\mathcal{C}$,*
  it can furthermore be interpreted as a continuous function.
- *A model of a theory $\mathbb{T}$ in the topos $\widehat{\mathcal{C}}_J$*
  can be seen as a continuous family of structures of type $\mathbb{T}$.

Sections of presheaves or sheaves

Definition. – *A* <u>*section*</u> *of a* <u>*presheaf*</u> *(or in particular of a sheaf)*

$$\begin{aligned} P : \mathcal{C} & \longrightarrow \mathrm{Set}, \\ X & \longmapsto P(X), \\ (X' \xrightarrow{x} X) & \longmapsto (P(X) \xrightarrow{P(x)} P(X')) \end{aligned}$$

is a <u>*family of elements*</u>

$$p_X \in P(X)\,, \quad X = \underline{\textit{element}} \textit{ of } \mathcal{C},$$

which is <u>*consistent*</u> *in the sense that*

$$p_{X'} \in P(x)(p_X)\,, \quad \forall\,(X' \xrightarrow{x} X) = \underline{\textit{arrow}} \textit{ of } \mathcal{C}.$$

Fact: If the sets $P(X)$ are <u>endowed with some structures</u> as

$$\begin{cases} - & \underline{\text{linear structure}} \text{ and } \underline{\text{scalar product}}, \\ - & \underline{\text{partial order structure}} \text{ with some property,} \end{cases}$$

some kind of combinatorial "<u>Laplace operator</u>"
allows to <u>transform arbitrary families</u> $(p_X \in P(X))_{X = \text{element of } \mathcal{C}}$
into <u>sections</u>, i.e. <u>consistent families</u>.

Remark: This provides a <u>computational model</u>
for <u>making consistent</u> families of <u>partial elements of information</u>.

Sheafification as a structure-valued averaging process

Theorem. – *Let J be a topology on a small category $\mathcal{C}$.*
(i) Then the embedding

$$\widehat{\mathcal{C}}_J \longrightarrow \widehat{\mathcal{C}}$$

has a well-defined "adjoint"

$$j^* : \widehat{\mathcal{C}} \longrightarrow \widehat{\mathcal{C}}_J$$

which transforms

presheaves, i.e. set-valued structured functions

into J-sheaves, i.e. makes these functions continuous.

(ii) Furthermore, for any (first-order geometric) theory $\mathbb{T}$, the functor

$$j^* : \widehat{\mathcal{C}} \longrightarrow \widehat{\mathcal{C}}_J$$

induces a functor

$$j^* : \mathbb{T}\text{-mod}(\widehat{\mathcal{C}}) \longrightarrow \mathbb{T}\text{-mod}(\widehat{\mathcal{C}}_J)$$

which makes arbitrary families of structures of type $\mathbb{T}$ continuous with respect to J.

Remark: This defines a geometric lifting
of the process of averaging functions (e.g. by convolution).