A mathematician's journey with Grothendieck

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Mathematics, history, literature and Latin

- At school and high-school,

  mathematics was an easy subject

  (later, it became more challenging)

  but I had other passions.

- Early passions:
  - History (already at elementary school age).
  - Literature (a little later):
    in particular French and Russian literature.
  - Latin language (especially at age 13-15).

Remarks:

- Mathematics has a very strong historical dimension.

  It is a collective tradition,

  built century after century, even millenia.

- In some sense, mathematics is a part of literature.
- Ancien Latin is a very structural language.

  Studying Latin was a good preparation for math. logic.

First discovery of Grothendieck's mathematics

- I think I had never heard about Grothendieck
  before my second year at "Ecole Normale Supérieure",
  at Master level.
- I discovered Grothendieck
  because an "initiation to algebraic geometry" seminar
  had been organized for students.
  I wondered what "algebraic geometry" could mean,
  and I decided to participate in this seminar.
- I quickly realized
  that algebraic geometry was a very old domain
  which had been given
  entirely new very modern foundations and methods
  by essentially one mathematician.
- Starting from this discovery moment,
  I spent several years studying EGA and SGA $\cdots$

Scheme theory

Difficult to say a few words about a continent $\cdots$
Two fascinating features of this continent were:

- The very definition of geometric objects (schemes)
  as topological spaces
  (of a strange kind: they were not Hausdorff!)
  endowed with a "sheaf of commutative rings"
  carrying the geometric structure.

  $\longrightarrow$ This idea came from Jean-Pierre Serre
  and had been lifted by Grothendieck
  to the natural level of generality.

- Schemes were not considered individually
  but through their relations with other schemes,
  formalized as a "category"
  endowed with powerful structures,
  particularly "fiber products" defining "base change":

$$\begin{tikzcd} S' \times_S X = X' \arrow[r, dashed] \arrow[d, dashed] & X \arrow[d] \\ S' \arrow[r] & S \end{tikzcd}$$

The fascinating notion of continuous family of geometric objects

- Before I discovered the theory of schemes,
  I only knew the notion of continuous map

  $S \longrightarrow X$ between topological spaces S, X,

  which could be considered as a notion of

  continuous family of elements of X
  indexed by elements of S.

- Then I discovered that, in scheme theory, a scheme morphism

  $$\begin{array}{c} X \\ \downarrow \\ S \end{array}$$

  which is both "flat" and "proper"
  (two perfectly defined properties)
  can be considered as a continuous family of

  algebraic varieties

  indexed by the points of S.

- Even more incredibly, if $S = \mathrm{Spec}(\mathbb{Z})$,
  points of S were prime numbers $2, 3, 5, 7, \cdots$

Discovering “Récoltes et Semailles” after EGA and SGA

- While I had studied EGA and SGA for several years
  and was preparing a PhD,
  I began reading “Récoltes et Semailles”:

  “Harvests and Sowings: Reflections and Testimonials
  on a Mathematician’s Past”.
- I discovered that Grothendieck was as singular as a man
  as his mathematics were singular.
- It seemed that nobody had loved mathematics as much as him,
  but he was also a writer, a poet and a thinker,
  and his love and passionate search for truth
  went far beyond mathematics.
- For him, mathematics consisted much more in definitions than in proofs.
- He was accusing well-known mathematicians
  and the “mathematical community” as a group.
- He was defending “orphans”: “orphan” theories
  and “orphan” people.

A most important example of orphan theory
(which has become mainstream 40 years later $\cdots$)

The "six operations" formalism:

This is Grothendieck's general theory of cohomology and cohomological duality

(which generalizes simultaneously

Poincaré duality in topology

and

Serre duality in geometry).

Ordinary inner operations:

$\otimes_{\mathcal{O}}, \mathcal{H}om_{\mathcal{O}}$ and their "derivations" $\overset{L}{\otimes}_{\mathcal{O}}, R\mathcal{H}om_{\mathcal{O}}$

well-defined over any topos $\mathcal{E}$

endowed with an inner commutative ring $\mathcal{O}$.

Ordinary outer operations:

f^*, f_* and their "derivations" Lf^*, Rf_*

well-defined for any topos morphism $\mathcal{E}' \xrightarrow{f} \mathcal{E}$

complete with a morphism $f^*\mathcal{O} \longrightarrow \mathcal{O}'$

relating inner commutative rings $\mathcal{O}, \mathcal{O}'$ on $\mathcal{E}, \mathcal{E}'$.

The "extraordinary operations" $f_!, f^!$

- They are associated to some topos morphisms (not all)
  $$\mathcal{E}' \xrightarrow{f} \mathcal{E}$$
  completed with a morphism $f^*\mathcal{O} \to \mathcal{O}'$.
- Mysteriously, they are usually well-defined
  in the context of "geometric categories"
  coming from classical or natural mathematical practice.
- They exist only at the level of "derived categories":
  $$\begin{tikzcd} D(\mathcal{M}od_{\mathcal{O}'}) \arrow[r, shift left, "f_!"] & D(\mathcal{M}od_{\mathcal{O}}) \arrow[l, shift left, "f^!"] \end{tikzcd}$$

One of the deepest drawings of mathematics:

$$\begin{tikzcd} D(\mathcal{M}od_{\mathcal{O}'})^{\mathrm{op}} \arrow[r, "R\mathcal{H}om_{\mathcal{O}'}(\bullet, f^!\mathcal{L})"] \arrow[d, "f_!"'] & D(\mathcal{M}od_{\mathcal{O}'}) \arrow[d, "Rf_*"] \\ D(\mathcal{M}od_{\mathcal{O}})^{\mathrm{op}} \arrow[r, "R\mathcal{H}om_{\mathcal{O}}(\bullet, \mathcal{L})"] & D(\mathcal{M}od_{\mathcal{O}}) \end{tikzcd}$$

This is "global duality" $\cdots$

An example of orphan person: Olivier Leroy

- He was a student in Montpellier of
  Carlos Couton-Carrère (a former student of Grothendieck).
- He worked on toposes.
- Grothendieck found him extraordinary talented
  but he never got any university position
  and eventually passed away at early age.
- After his death,
  some mathematical manuscripts
  were found in his belongings.
- In one of his manuscripts,
  he had shown that
  the notion of subtopos
  solves the famous
  Banach-Tarski paradox.

Banach-Tarski paradox. –

For a measure μ of open subsets U of a space X
which is invariant by a symmetry group G,
it is not possible in general to extend μ to all subsets $X' \hookrightarrow X$
in a way which respects both G-invariance and additivity

$$\mu(X_1') + \mu(X_2') = \mu(X_1' \cup X_2') + \mu(X_1' \cap X_2') \,.$$

Leroy's solution. –

There exists a natural extension of μ to all subtoposes

$$\mathcal{E}' \hookrightarrow \mathcal{E}_X \qquad \textit{of} \quad \mathcal{E}_X = \textit{topos defined by } X$$

which respects both G-invariance and additivity.
This measure restricts in particular to subsets $\quad X' \hookrightarrow X$
as any subset $\quad X' \hookrightarrow X \quad$ *defines a subtopos*

$$\mathcal{E}_{X'} \longrightarrow \mathcal{E}_X \,.$$

Explanation:

In general, one has $\qquad \mathcal{E}_{X_1' \cap X_2'} \neq \mathcal{E}_{X_2'} \cap \mathcal{E}_{X_2'}$.
It may even happen that

$$X_1' \cap X_2' = \emptyset \quad \underline{\text{but}} \quad \mu(\mathcal{E}_{X_1'} \cap \mathcal{E}_{X_2'}) > 0 \,.$$

From Grothendieck's geometry to the Langlands program

- Beginning with my PhD thesis under Prof. Gérard Laumon,
  I made use of Grothendieck's powerful theories,
  following Drinfeld, to ultimately prove
  Langland's correspondence for groups GL_r over
  function fields F_C of smooth projective curves C over finite fields $\mathbb{F}_q$.

Main ingredients:

- Consider the classification problem of Drinfeld's rank r "shtukas"
  over such a curve C.
- Show this classification problem has a solution in Grothendieck's sense,
  as a "classifying" algebraic stack Sht_C^r.
- Consider the cohomology spaces (with ℓ-adic coefficients)
  of the "étale toposes" associated to these algebraic stacks Sht_C^r,
  with the structural double actions on them of
  Galois groups $\mathcal{G}al(F_C)$ and linear arithmetic groups $\mathrm{GL}_r(\mathbb{A}_{F_C})$.
- Use tools from Grothendieck's cohomological duality theory,
  especially the "Grothendieck-Lefschetz fixed points formula",
  to show that the ℓ-adic cohomology spaces of Sht_C^r
  endowed with the natural actions of $\mathcal{G}al(F_C)$ and $\mathrm{GL}_r(\mathbb{A}_{F_C})$
  realize Langlands' correspondence.

What is a "classification problem" in Grothendieck's sense?

Studying Grothendieck, I discovered that, for him,
a well-posed classification problem
consists in a definition of a general notion of
"continuous family of structures of some type $\mathbb{T}$
indexed by points of spaces S of some type $\mathcal{C}$"
such that:

(1) For any space S of the type $\mathcal{C}$,
continuous families of structures of type $\mathbb{T}$ induced by S
make up a set

$M_{\mathbb{T}}(S)$ (or more generally a "category" $\mathcal{M}_{\mathbb{T}}(S)$).

(2) For any map between spaces S, S' of type $\mathcal{C}$

$S' \xrightarrow{s} S$ which respects their geometric structures,

there exists an associated "change of parametrization" map

$$s^* : M_{\mathbb{T}}(S) \longrightarrow M_{\mathbb{T}}(S')$$

(or more generally a functor $s^* : \mathcal{M}_{\mathbb{T}}(S) \to \mathcal{M}_{\mathbb{T}}(S')$).

What is a "classifying space" in Grothendieck's sense?

Definition. –

Consider a <u>*classification problem*</u> *in* <u>*Grothendieck's sense*</u>

$$\begin{array}{ccc} \mathcal{C} & \longrightarrow & \mathrm{Set}\ (\textit{or}\ \mathrm{Cat}), \\ \left.\begin{array}{c}\underline{\textit{space}}\ S \\ \textit{of}\ \mathcal{C}\end{array}\right\} & \longmapsto & \begin{cases} M_{\mathbb{T}}(S) = \underline{\textit{set}}, \textit{ or} \\ \mathcal{M}_{\mathbb{T}}(S) = \underline{\textit{category}}, \end{cases} \\ \left.\begin{array}{c}\underline{\textit{geometric map}}\ S' \xrightarrow{s} S \\ \textit{of}\ \mathcal{C}\end{array}\right\} & \longmapsto & \begin{cases} M_{\mathbb{T}}(S) \xrightarrow{s^*} M_{\mathbb{T}}(S'), \textit{ or} \\ \mathcal{M}_{\mathbb{T}}(S) \xrightarrow{s^*} \mathcal{M}_{\mathbb{T}}(S'). \end{cases} \end{array}$$

Then a (necessarily <u>*unique*</u>*) "*<u>*classifying space*</u>*" for this* <u>*classification problem*</u>

is a <u>*space*</u> $\quad S_{\mathbb{T}} \quad$ *in* $\quad \mathcal{C}$

<u>*endowed*</u> *with a so-called*

<u>*universal continuous family*</u> *indexed by* $S_{\mathbb{T}}$

$$U_{\mathbb{T}} \quad \textit{in} \quad M_{\mathbb{T}}(S_{\mathbb{T}}) \quad \textit{or} \quad \mathcal{M}_{\mathbb{T}}(S_{\mathbb{T}}),$$

such that, for <u>*any space*</u> *S of $\mathcal{C}$, the* <u>*map*</u> *(or* <u>*functor*</u>*)*

$$\begin{array}{ccc} \left\{\begin{array}{c}\underline{\textit{geometric maps}} \\ S \to S_{\mathbb{T}}\ \textit{in}\ \mathcal{C}\end{array}\right\} & \longrightarrow & M_{\mathbb{T}}(S) \quad \textit{or} \quad \mathcal{M}_{\mathbb{T}}(S)\,, \\ (S \xrightarrow{s} S_{\mathbb{T}}) & \longmapsto & s^* \, U_{\mathbb{T}} \end{array}$$

is <u>*one-to-one*</u> *(or is an* <u>*equivalence*</u>*).*

Classifying toposes

In 2010, I learnt from a young post-doc, Olivia Caramello, the following theorem which had been proved in the mid 1970's by Makkai and Reyes, building on previous work of Grothendieck, Monique Hakim, Joyal, $\cdots$

Theorem. –
Consider an arbitrary "first-order theory" $\mathbb{T}$ presented in "geometric form".
Then:

(i) *For any "topos" $\mathcal{E}$ (defined by Grothendieck as the most general notion of space), continuous families of structures of type $\mathbb{T}$ indexed by $\mathcal{E}$*

make up a category $\qquad \mathcal{M}_{\mathbb{T}}(\mathcal{E})$.

(ii) *Any topos "morphism"* $\quad \mathcal{E}' \xrightarrow{f} \mathcal{E}$
(defined by Grothendieck as the natural generalization of continuous maps) defines a functor of "change of parametrization" of continuous families

$$f^* : \mathcal{M}_{\mathbb{T}}(\mathcal{E}) \longrightarrow \mathcal{M}_{\mathbb{T}}(\mathcal{E}').$$

(iii) *Whatever $\mathbb{T}$, this classification problem defines*

a "classifying topos" $\ \mathcal{E}_{\mathbb{T}}$

which is uniquely determined up to equivalence.

Remark:
For any such a theory $\mathbb{T}$, i.e. any formal language (of "syntactic" nature), $\mathcal{E}_{\mathbb{T}}$ incarnates its "semantics" as a generalized space.

Caramello's technique of "toposes as bridges"

Basic facts. –

(i) *Following Grothendieck, any "geometric datum" consisting in*

$$\begin{cases} \bullet & \textit{a category } \mathcal{C}\textit{, i.e. a world of spaces of some type,} \\ \bullet & \textit{a "topology" (in a generalized sense) } J \textit{ on } \mathcal{C} \end{cases}$$

defines a topos $\mathcal{E} = \widehat{\mathcal{C}}_J$.
Conversely, any topos $\mathcal{E}$ *is associated to infinitely diverse geometric data* $(\mathcal{C}, J)$.

(ii) *Any formal language* $\mathbb{T}$ *of "first-order geometric" type defines a "classifying topos" which incarnates its semantics.*
Conversely, any topos $\mathcal{E}$ *is equivalent to the classifying topos* $\mathcal{E}_{\mathbb{T}}$ *of infinitely diverse formal languages* $\mathbb{T}$.

Principles for the "bridge technique". –

(1) *Consider invariants of toposes.*

(2) *Express them or compute them in terms of different types of presentation by geometric data* $(\mathcal{C}, J)$ *or by formal languages* $\mathbb{T}$.

(3) *Combine these expressions or computations with equivalences of toposes associated to concrete situations.*

An example of the "bridge technique"

Basic geometric fact. –
Grothendieck has defined a notion of "subtopos" of a topos $\mathcal{E}$

$$\mathcal{E}' \hookrightarrow \mathcal{E}$$

which generalizes the notion of subspace of a space.

Theorem (Grothendieck). –
For any presentation of a topos $\mathcal{E}$ *by a geometric datum* $(\mathcal{C}, J)$

$$\mathcal{E} \cong \widehat{\mathcal{C}}_J\,,$$

subtoposes $\quad \mathcal{E}' \hookrightarrow \mathcal{E}$
have a one-to-one correspondence with
topologies J' *on the underlying category* $\mathcal{C}$ *which refine* J.

Theorem (Caramello). –
For any presentation of a topos $\mathcal{E}$ *as classifying a theory* $\mathbb{T}$

$$\mathcal{E} \cong \mathcal{E}_{\mathbb{T}}\,,$$

subtoposes $\quad \mathcal{E}' \hookrightarrow \mathcal{E}$
have a one-to-one correspondence with

$\begin{cases} \textit{systems of extra axioms added to } \mathbb{T}, \\ \textit{considered up to provable equivalence.} \end{cases}$

From Grothendieck's geometry to AI following toposes

Application to provability problems. – *The double equivalence*

$$\left\{\begin{array}{c}\underline{\textit{topologies}}\\ \textit{on categories}\end{array}\right\} \leftrightarrow \left\{\begin{array}{c}\underline{\textit{subtoposes}}\\ \textit{of a topos}\end{array}\right\} \leftrightarrow \left\{\begin{array}{c}\underline{\textit{systems of extra axioms,}}\\ \textit{considered } \underline{\textit{up to provable equivalence}}\end{array}\right\}$$

is <u>constructive</u> and allows <u>computation</u>.
It allows to <u>transform</u> any <u>provability problem</u> in math
into <u>problems</u> of <u>generating topologies</u>.

Application to "syntactic learning". –
The definition and construction of <u>classifying toposes</u> $\mathcal{E}_{\mathbb{T}}$
relates in a perfectly precise and rigorous way

- *the problem of <u>describing</u> and <u>representing</u> <u>any type of data</u> we are interested in,*
- *the problem of <u>classifying data</u> of any type.*

Application to the possible creation of a theory of semantic information. –
Such a theory could and should be based on the <u>construction</u>

$$\mathbb{T} \longmapsto \mathcal{E}_{\mathbb{T}}\,.$$

Application to "meta-learning". – *The notion of <u>relative topos</u>* $\quad \mathcal{E}' \to \mathcal{E}$
provides a <u>mathematical model</u> for "<u>meta-learning</u>",
i.e. extracting information at <u>higher levels of abstraction</u>.

Helping to create and develop the "Grothendieck Institute"

- The "Grothendieck Institute"
  is a Scientific Foundation.
  It was created in 2022 by Prof. Olivia Caramello
  and is recognized by the Italian government.

Purposes of the Grothendieck Institute. –

(1) *To disseminate the knowledge of Grothendieck's*
   - *published work,*
   - *manuscripts,*
   - *thought.*

(2) *To contribute to the continuous development of*
   - *mathematics inherited from Grothendieck,*
   - *general unifying theories such as those he introduced,*
   - *in particular Topos Theory and its applications.*

Remark:

Grothendieck's thought goes beyond mathematics.

For instance, his book

"La Clef des Songes" ("The Key of Dreams")

has deep philosophical content.