Topological translation of first-order logic

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First-order languages

Definition. –

A "first-order language" Σ consists of

- *a set of "object names" or "sorts",*
- *a set of "relation names (or symbols)",*

  $$R \rightarrowtail A_1 \cdots A_n$$

  in the context of a finite sequence of sorts $A_1, \cdots, A_n$, $n \geq 0$,
- *a set of "function names (or symbols)",*

  $$f : A_1 \cdots A_n \longrightarrow B$$

  over a finite sequence of sorts $A_1, \cdots, A_n$, $n \geq 0$,

  with values in a sort B.

Remark:
Such a language is called

- "relational" if it has no function names,
- "algebraic" if it has no relation names,
- "propositional" if it has no sorts, and hence only relation names $R \rightarrowtail \quad$.

First-order formulas and their formation

Definition. –

(i) *A "context" is a finite sequence of variables* $\vec{x} = (x_1^{A_1}, \cdots, x_n^{A_n})$ *associated with sorts* $A_1, \cdots, A_n$ *of the language* Σ.

(ii) *A formula* $\varphi(\vec{x})$ *in a context* $\vec{x}$ *can be*

- *a relation* $R(x_1^{A_1}, \cdots, x_n^{A_n})$ *among some of the variables of* $\vec{x}$, *for a relation symbol* $R \rightarrowtail A_1 \cdots A_n$ *of* Σ,
- *an equality* $x^B = y^B$ *between two variables of* $\vec{x}$ *of the same sort,*
- *derived from already defined formulas by forming*

$\left\{ \begin{array}{l} \text{– } \textit{substitutions } x^B = f(x_1^{A_1}, \cdots, x_n^{A_n}) \textit{ of variables by function symbols} \\ \quad f : A_1 \cdots A_n \longrightarrow B \textit{ applied to variables,} \\ \text{– } \textit{a finite conjunction } \wedge \textit{ or infinite conjunction } \bigwedge \\ \quad \textit{of formulas } \varphi_i(\vec{x}), i \in I, \textit{ in the same context } \vec{x}, \textit{ and the "true" symbol } \top(\vec{x}) \textit{ in the context } \vec{x}, \\ \text{– } \textit{a finite disjunction } \vee \textit{ or infinite disjunction } \bigvee \\ \quad \textit{of formulas } \varphi_i(\vec{x}), i \in I, \textit{ in the same context } \vec{x}, \textit{ and the "false" symbol } \perp(\vec{x}) \textit{ in the context } \vec{x}, \\ \text{– } \textit{existential or universal quantifications over other variables} \\ \qquad\qquad (\exists \vec{y})\, \varphi(\vec{x}, \vec{y}) \quad \textit{or} \quad (\forall \vec{y})\, \varphi(\vec{x}, \vec{y}), \\ \text{– } \textit{the negation of a formula or the implication between two formulas} \quad \neg\, \varphi(\vec{x}) \quad \textit{or} \quad (\varphi(\vec{x}) \Rightarrow \psi(\vec{x})) \,. \end{array} \right.$

A brief reminder on the interpretation of formulas

Definition. –
A "model" of a language Σ in a topos $\mathcal{E}$ (such as $\mathcal{E} = \textbf{Sets}$) is a map

$$\begin{cases} \underline{\textit{sort}}\ A \textit{ of } \Sigma & \longmapsto & \underline{\textit{object}}\ MA \textit{ of } \mathcal{E}, \\ \underline{\textit{relation name}}\ (R \rightarrowtail A_1 \cdots A_n) & \longmapsto & \underline{\textit{subobject}}\ MR \hookrightarrow MA_1 \times \cdots \times MA_n, \\ \underline{\textit{function name}}\ (f : A_1 \cdots A_n \to B) & \longmapsto & \underline{\textit{morphism}}\ MA_1 \times \cdots \times MA_n \xrightarrow{Mf} MB. \end{cases}$$

Principle.–
Every formula $\varphi(\vec{x})$ in a context $\vec{x} = (x_1^{A_1}, \cdots, x_n^{A_n})$ defines
a subobject $\quad M\varphi \hookrightarrow MA_1 \times \cdots \times MA_n$.

Formation rules for subobjects defined by formulas:

- substitutions $x^B = f(x_1^{A_1}, \cdots, x_n^{A_n})$ define the inverse images of subobjects along the morphism $Mf : MA_1 \times \cdots \times MA_n \to MB$,
- conjunctions define intersections of subobjects,
- disjunctions define unions of subobjects,
- quantifications $(\exists\, \vec{y})$ and $(\forall\, \vec{y})$ over $(\vec{x}, \vec{y}) = (x_1^{A_1}, \cdots, x_n^{A_n}, y_1^{B_1}, \cdots, y_m^{B_m})$ define the direct images p_* and $p_!$ via the right and left adjoint functors to the inverse image functor p^{-1} along
$$p : MA_1 \times \cdots \times MA_n \times MB_1 \times \cdots \times MB_m \longrightarrow MA_1 \times \cdots \times MA_n\,,$$
- negations $\neg\varphi$ and implications $(\varphi \Rightarrow \psi)$ are interpreted as
$$\max\{S \mid S \cap M\varphi = \emptyset\} \quad \text{and} \quad \max\{S \mid S \cap M\varphi \subseteq M\psi\}\,.$$

Geometric formulas

Definition. – *A topos morphism* $\mathcal{E}' \xrightarrow{e} \mathcal{E}$
is a pair of adjoint functors $(\mathcal{E} \xrightarrow{e^*} \mathcal{E}', \mathcal{E}' \xrightarrow{e_*} \mathcal{E})$
whose component $e^* : \mathcal{E} \to \mathcal{E}'$ *preserves not only colimits but also finite limits.*

Proposition. –

(i) *The inverse image components* $e^* : \mathcal{E} \to \mathcal{E}'$ *of topos morphisms* $e : \mathcal{E}' \to \mathcal{E}$ *transform*

objects of $\mathcal{E}$	*into*	*objects of* $\mathcal{E}'$,
finite products of objects	*into*	*finite products,*
morphisms	*into*	*morphisms,*
subobjects	*into*	*subobjects.*

(ii) *They preserve*

- *inverse images of subobjects along morphisms,*
- *finite intersections (but not infinite intersections in general),*
- *arbitrary unions,*
- *the interpretations* p_* *of existential quantifier symbols* $\exists$
  (but not the interpretations $p_!$ *of universal quantifier symbols* $\forall$ *in general,*
  nor the interpretations of negations $\neg$ *and implications* $\Rightarrow$*).*

Definition. –
A formula is called "geometric" if its formation uses only the symbols $\wedge, \top, \bigvee, \bot, \exists$
and not the symbols $\bigwedge, \forall, \neg, \Rightarrow$.

First-order theories

Definition. –

(i) *A* <u>*first-order theory*</u> $\mathbb{T}$ *consists of*

- *a* <u>*first-order language*</u> Σ,
- *a* <u>*family of axioms*</u> *which are "*<u>*sequents*</u>*" (or "*<u>*implications*</u>*")*

  $\varphi_i(\vec{x}_i) \vdash \psi_i(\vec{x}_i)$ *between* <u>*formulas*</u> *in the* <u>*same context*</u> $\vec{x}_i$.

(ii) *A "*<u>*sequent*</u>*" relating two formulas in the* <u>*same context*</u>

$$\varphi(\vec{x}) \vdash \psi(\vec{x})$$

is said to be "<u>*provable*</u>*" in* $\mathbb{T}$ *if it is derived from the* <u>*axioms*</u>
by the "<u>*inference rules of first-order logic*</u>*".*

Definition. –

A <u>*first-order theory*</u> $\mathbb{T}$ *is called "*<u>*geometric*</u>*"*
if all its axioms

$$\varphi_i(\vec{x}_i) \vdash \psi_i(\vec{x}_i)$$

relate <u>*formulas*</u> φ_i, ψ_i *that are "*<u>*geometric*</u>*"*
(and whose interpretations are therefore <u>*preserved*</u>
by the <u>*inverse image components*</u> $e^* : \mathcal{E} \to \mathcal{E}'$
of <u>*topos morphisms*</u> $e : \mathcal{E}' \to \mathcal{E}$*).*

Semantics of first-order theories

Definition. –

Let Σ be a first-order language.

Let

$\varphi(\vec{x}) \vdash \psi(\vec{x})$ *be a "sequent" relating two formulas of Σ in the same context $\vec{x} = (x_1^{A_1}, \cdots, x_n^{A_n})$.*

We say that a model M of Σ in a topos $\mathcal{E}$

satisfies the sequent $\varphi \vdash \psi$

if the two subobjects $M\varphi$ and $M\psi$ of $MA_1 \times \cdots \times MA_n$ defined by φ and ψ

satisfy the inclusion relation $M\varphi \subseteq M\psi$.

Definition. –

A model M in a topos $\mathcal{E}$ of a theory $\mathbb{T}$ of language Σ

is a model M of Σ in $\mathcal{E}$

that satisfies all axioms $\varphi_i \vdash \psi_i$ that define $\mathbb{T}$.

Proposition. –

Every model M of a theory $\mathbb{T}$ of language Σ

satisfies every sequent $\varphi(\vec{x}) \vdash \psi(\vec{x})$ of Σ that is provable in $\mathbb{T}$.

The existence theorem for classifying toposes

Successive work by Monique Hakim (with Grothendieck), Lawvere, Joyal, $\cdots$ culminated in:

Theorem (Makkai, Reyes, 1975). –

Let $\mathbb{T}$ be a geometric first-order theory.
Then there exists a topos $\mathcal{E}_{\mathbb{T}}$ (called the classifying topos of $\mathbb{T}$) equipped with a model $U_{\mathbb{T}}$ of $\mathbb{T}$ (called the universal model of $\mathbb{T}$) such that, for every topos $\mathcal{E}$, the functor

$$\begin{array}{ccc} \mathcal{G}eom(\mathcal{E},\mathcal{E}_{\mathbb{T}}) & \longrightarrow & \mathbb{T}\text{-}\mathcal{M}od(\mathcal{E}) \\ \| & & \| \\ \left\{\begin{array}{c}\textit{category of topos morphisms} \\ e:\mathcal{E}\to\mathcal{E}_{\mathbb{T}}\end{array}\right\} & & \left\{\begin{array}{c}\textit{category of models } M \\ \textit{of } \mathbb{T} \textit{ in } \mathcal{E}\end{array}\right\} \\ & & \\ e & \longmapsto & e^* U_{\mathbb{T}} \end{array}$$

is an equivalence of categories.

Theorem. –

If $\mathbb{T}$ is a geometric theory of language Σ, a geometric sequent $\varphi(\vec{x}) \vdash \psi(\vec{x})$ is provable in $\mathbb{T}$ if and only if it is satisfied by the model $U_{\mathbb{T}}$ of $\mathbb{T}$ in $\mathcal{E}_{\mathbb{T}}$.

Reduction to geometric theories

Definition (Morley). –
To every first-order theory $\mathbb{T}$ of language Σ and axioms

$$\varphi_i(\vec{x}_i) \vdash \psi_i(\vec{x}_i)\,, \quad i \in I\,,$$

one associates the geometric first-order theory $\mathbb{T}_{\mathcal{M}}$,
whose language $\Sigma_{\mathcal{M}}$ is the language Σ extended by relation symbols,
constructed as follows:

(1) *Each time that in the construction of the φ_i, ψ_i, $i \in I$,*
an implication $\quad \varphi(\vec{x}) \Rightarrow \psi(\vec{x}) \quad$ *appears,*
it is replaced by the formula $\quad (\neg\varphi(\vec{x})) \vee \psi(\vec{x})$.

(2) *Each time that in this construction,*
a universal quantification $\quad (\forall \vec{y})\,\varphi(\vec{x},\vec{y}) \quad$ *appears,*
it is replaced by the formula $\quad \neg(\exists\,\vec{y})(\neg\varphi(\vec{x},\vec{y}))$.

(3) *Once these substitutions are made, each time a negation*

$$\neg\varphi(\vec{x}) \quad \textit{of a formula } \varphi \textit{ in context } \vec{x} = (x_1^{A_1}, \cdots, x_n^{A_n})$$

appears in the construction of the rewritten formulas composing the axioms,
it is replaced by a new relation symbol

$$N_\varphi \rightarrowtail A_1 \cdots A_n$$

subject to the additional axioms

$$\varphi(\vec{x}) \wedge N_\varphi(\vec{x}) \vdash \bot \quad \textit{and} \quad \top \vdash \varphi(\vec{x}) \vee N_\varphi(\vec{x}).$$

Effect of geometric reductions on the semantics of theories

Proposition. –

Let $\mathbb{T}$ be a first-order theory.
Let $\mathbb{T}_{\mathcal{M}}$ be the theory derived from it by the Morley process.
Then:

(i) *For every topos $\mathcal{E}$, there is a canonical functor*

$$\mathbb{T}_{\mathcal{M}}\text{-}\mathcal{M}od(\mathcal{E}) \longrightarrow \mathbb{T}\text{-}\mathcal{M}od(\mathcal{E})$$

that is injective both on objects and on morphisms.

(ii) *If $\mathcal{E}$ is the topos of sets*
or more generally a "Boolean topos",
the canonical functor

$$\mathbb{T}_{\mathcal{M}}\text{-}\mathcal{M}od(\mathcal{E}) \longrightarrow \mathbb{T}\text{-}\mathcal{M}od(\mathcal{E})$$

is bijective on objects
(but not necessarily on morphisms).

Remark:

A topos is "Boolean" if, for every subobject $S \hookrightarrow E$ of an object E,
its complement $E \backslash S = \max\{S' \hookrightarrow E \mid S' \cap S = \emptyset\}$
satisfies $(E \backslash S) \cup S = E$.

The presentation problem for classifying toposes

Theorem. – *The classifying topos of a geometric first-order theory* $\mathbb{T}$ *can be presented as the topos of sheaves*

$$\mathcal{E}_{\mathbb{T}} = \widehat{(\mathcal{C}_{\mathbb{T}})}_{J_{\mathbb{T}}}$$

on the geometric syntactic category $\mathcal{C}_{\mathbb{T}}$ *of* $\mathbb{T}$ *equipped with its canonical topology* $J_{\mathbb{T}}$.

Definition. – *If* $\mathbb{T}$ *is a geometric first-order theory of language* Σ, *its syntactic category* $\mathcal{C}_{\mathbb{T}}$ *is defined as follows:*

(i) *The objects of* $\mathcal{C}_{\mathbb{T}}$ *are the geometric formulas of* Σ

$$\varphi(\vec{x}) \quad \text{in contexts } \vec{x} = (x_1^{A_1}, \cdots, x_n^{A_n})$$

up to renaming of variables.

(ii) *The morphisms of* $\mathcal{C}_{\mathbb{T}}$ *are the geometric formulas*

$$\varphi(\vec{x}) \xrightarrow{\theta(\vec{x},\vec{y})} \psi(\vec{y})$$

that are "provably functional" in $\mathbb{T}$.

(iii) *The "canonical topology" of* $\mathcal{C}_{\mathbb{T}}$ *is the one for which a family of morphisms*

$$\theta_i(\vec{x}_i, \vec{y}) : \varphi_i(\vec{x}_i) \longrightarrow \psi(\vec{y}), \quad i \in I,$$

is "covering" if and only if the sequent

$$\psi(\vec{y}) \vdash \bigvee_{i \in I} (\exists \vec{x}_i) \theta_i(\vec{x}_i, \vec{y}) \quad \text{is provable in } \mathbb{T}.$$

Grothendieck topologies and subtoposes

Definition. – *A Grothendieck topology J on a category $\mathcal{C}$ is a notion of "covering presieve" on objects X of $\mathcal{C}$ by families of morphisms*

$$(X_i \xrightarrow{x_i} X)_{i \in I}$$

such that:

(0) *(Identity) For every object X, $X \xrightarrow{\text{id}} X$ is covering.*

(1) *(Composition) If $(X_i \xrightarrow{x_i} X)_{i \in I}$ is J-covering and, for every $i \in I$, $(X_{i,j} \xrightarrow{x_{i,j}} X_i)_{j \in I_i}$ is J-covering, then $(X_{i,j} \xrightarrow{x_i \circ x_{i,j}} X)_{i \in I, j \in I_i}$ is J-covering.*

(2) *(Factorisation) If $(X_i \xrightarrow{x_i} X)_{i \in I}$ is J-covering, and if each $X_i \xrightarrow{x_i} X$ factors through some $X'_{i'} \xrightarrow{x'_{i'}} X$, $i' \in I'$, then the presieve $(X'_{i'} \xrightarrow{x'_{i'}} X)_{i' \in I'}$ is J-covering.*

(3) *(Induction) For every morphism $X' \xrightarrow{x} X$ and every J-covering presieve $(X_i \xrightarrow{x_i} X)_{i \in I}$, there exists a J-covering presieve $(X'_{i'} \xrightarrow{x'_{i'}} X')_{i' \in I'}$ such that*

$$\begin{cases} \bullet \quad \text{each composite morphism } x \circ x'_{i'} : X'_{i'} \to X' \to X \\ \quad\ \ \text{factors through one of the } X_i \xrightarrow{x_i} X. \end{cases}$$

Theorem (Grothendieck). – *There exists an intrinsic notion of subtopos of a topos $\mathcal{E}$ such that, if $\mathcal{E} \cong \widehat{\mathcal{C}}_J$ is presented as the topos of J-sheaves on a category $\mathcal{C}$, the subtoposes of $\mathcal{E}$ correspond to the topologies J' on $\mathcal{C}$ that refine J.*

Subtoposes and quotient theories

Theorem (Caramello). –

Let $\mathbb{T}$ be a geometric first-order theory of language Σ.

(i) *Every "quotient theory" $\mathbb{T}'$ of $\mathbb{T}$,*
that is, a theory of the same language Σ obtained from $\mathbb{T}$ by adding axioms,
has as its classifying topos a subtopos of $\mathcal{E}_{\mathbb{T}}$

$$\mathcal{E}_{\mathbb{T}'} \hookrightarrow \mathcal{E}_{\mathbb{T}}.$$

(ii) *Two quotient theories $\mathbb{T}_1$ and $\mathbb{T}_2$ of $\mathbb{T}$*
define the same subtopos $\mathcal{E}_{\mathbb{T}_1} \cong \mathcal{E}_{\mathbb{T}_2} \hookrightarrow \mathcal{E}_{\mathbb{T}}$
if and only if they are "syntactically equivalent"
in the sense that the axioms of each are provable from the axioms of the other.

(iii) *Every subtopos of $\mathcal{E}_{\mathbb{T}}$*
is the classifying topos of a quotient theory $\mathbb{T}'$ of $\mathbb{T}$.

Consequence:

There is a constructive equivalence between

- provability problems in geometric first-order logic,
- generation problems for Grothendieck topologies.

Cartesian theories and cartesian syntactic categories

Definition. –

(i) *A geometric formula of a language* Σ *is called*

- *"atomic" if it is derived from a formula of the form*
  $R(x_1^{A_1}, \cdots, x_n^{A_n})$ *for a relation symbol* $R \rightarrowtail A_1 \cdots A_n$ *of* Σ
  or $x_1^B = x_2^B$ *for a sort* B *of* Σ*,*
  by substitution a finite number of times of variables y^B *by*
  $y^B = f(x_1^{B_1}, \cdots, x_m^{B_m})$ *for function symbols* $f : B_1 \cdots B_m \longrightarrow B$ *of* Σ*,*
- *"Horn" if it is a finite conjunction of "atomic formulas".*

(ii) *A formula of* Σ *is called "*$\mathbb{T}$*-cartesian" for a theory* $\mathbb{T}$ *of language* Σ
if it has the form

$$(\exists \vec{y})\, \varphi(\vec{x}, \vec{y})$$

for a "Horn formula" $\varphi(\vec{x}, \vec{y})$ *such that the sequent*

$$\varphi(\vec{x}, \vec{y}) \wedge \varphi(\vec{x}, \vec{y}\,') \vdash \vec{y} = \vec{y}\,' \quad \text{is provable in } \mathbb{T}.$$

(iii) *A theory* $\mathbb{T}$ *of language* Σ *is called cartesian*
if it can be defined by a family of axioms

$$\varphi_i(\vec{x}_i) \vdash \psi_i(\vec{x}_i)\,, \quad i \in I\,,$$

consisting of $\mathbb{T}$*-cartesian formulas* φ_i, ψ_i*,* $i \in I$*.*

Definition. – *If* $\mathbb{T}$ *is a cartesian theory, its cartesian syntactic category is the subcategory* $\mathcal{C}_{\mathbb{T}}^c \hookrightarrow \mathcal{C}_{\mathbb{T}}$
consisting of objects $\varphi(\vec{x})$ *and morphisms* $\theta(\vec{x}, \vec{y}) : \varphi(\vec{x}) \vdash \psi(\vec{y})$ *that are* $\mathbb{T}$*-cartesian formulas.*

Representation via cartesian syntactic categories

Theorem. –
Let $\mathbb{T}$ be a geometric theory that is cartesian.
Then its classifying topos is presented as a presheaf topos

$$\mathcal{E}_{\mathbb{T}} \cong \widehat{\mathcal{C}^c_{\mathbb{T}}}$$

on the cartesian syntactic category of $\mathbb{T}$.

Remarks:

(i) In particular, every theory without axioms,
that is, reduced to a single language Σ, is cartesian
and we have

$$\mathcal{E}_{\Sigma} \cong \widehat{\mathcal{C}^c_{\Sigma}}.$$

(ii) Every algebraic theory is also cartesian.

(iii) More generally, a theory $\mathbb{T}$ is said to be of "presheaf type"
if its classifying topos is presented as a presheaf topos

$$\mathcal{E}_{\mathbb{T}} \cong \widehat{\mathcal{C}} \quad \text{on a category } \mathcal{C}.$$

One shows that, if $\mathbb{T}$ is of presheaf type, we always have

$$\mathcal{E}_{\mathbb{T}} \cong \widehat{\mathcal{M}^{\mathrm{op}}}$$

where $\mathcal{M}$ denotes the full subcategory

$$\mathcal{M} \hookrightarrow \mathbb{T}\text{-}\mathcal{M}od(\mathrm{Set})$$

of set-theoretic models of $\mathbb{T}$
that are "finitely presentable" by a finite family of generators and relations.

The computability problem in categories

Observation. – *Every geometric first-order theory $\mathbb{T}$ can be presented, in general in multiple ways, as a "quotient theory" of a cartesian theory $\mathbb{T}_c$.*

Explanation: One can take as the cartesian theory $\mathbb{T}_c$

- the theory without axioms reduced to the language Σ of $\mathbb{T}$,
- any theory of language Σ defined by a family of $\mathbb{T}$-cartesian axioms provable in $\mathbb{T}$.

Consequences:

- *The classifying topos $\mathcal{E}_{\mathbb{T}}$ is a subtopos of $\mathcal{E}_{\mathbb{T}_c} \cong \widehat{\mathcal{C}^c_{\mathbb{T}_c}}$.*
- *We have $\mathcal{E}_{\mathbb{T}} \cong \widehat{(\mathcal{C}^c_{\mathbb{T}_c})}_{J_{\mathbb{T}}}$ for a topology $J_{\mathbb{T}}$ on $\mathcal{C}^c_{\mathbb{T}}$ generated by covering presieves constructed from the axioms of $\mathbb{T}$.*
- *Every provability problem in $\mathbb{T}$ amounts to asking whether certain presieves of $\mathcal{C}^c_{\mathbb{T}_c}$ are covering for the generated topology $J_{\mathbb{T}}$.*

Problem.–
In general, the sets of morphisms between two objects of the categories

$$\mathcal{C}^c_{\mathbb{T}_c} = \mathcal{M}^{\mathrm{op}} \qquad (\text{where } \mathcal{M} = \mathbb{T}_c\text{-}\mathcal{M}od(\mathrm{Set})_{\mathrm{fp}})$$

are not computable,
and the factorisability problem for a morphism through another is too difficult.

Local computability for relational languages

Proposition. – *Consider a language Σ that is "relational", that is, without function symbols, reduced to sorts and relation symbols.*
Then the cartesian syntactic category $\mathcal{C}^c_\Sigma$ of Σ is described as follows:

(i) *Its objects are indexed by finite combinatorial data consisting of*

- *a finite set $\mathcal{A}$ of sorts of Σ,*
- *multiplicities $m_A \geq 1$, $A \in \mathcal{A}$,*
- *a finite set $\mathcal{R}$ of pairs (R, α) consisting of*
  - *a relation symbol $R \rightarrowtail A_1 \cdots A_{n_R}$ of Σ,*
  - *a finite sequence $\alpha = \Big(\alpha(i) \in \{1, 2, \cdots, m_{A_i}\}\Big)_{1 \leq i \leq n_R}$.*

(ii) *The morphisms from an object indexed by $(\mathcal{A}, m_\bullet, \mathcal{R})$ to an object indexed by $(\mathcal{A}', m'_\bullet, \mathcal{R}')$ are indexed by the families of maps*

$$f_A : \{1, \cdots, m'_A\} \longrightarrow \{1, \cdots, m_A\}, \quad A \in \mathcal{A}' \subseteq \mathcal{A},$$

such that, for every pair $(R \rightarrowtail A_1 \cdots A_n, \beta) \in \mathcal{R}'$, the pair $(R \rightarrowtail A_1 \cdots A_n, \alpha)$ defined by $\alpha(i) = f_{A_i}(\beta(i))$, $1 \leq i \leq n_R$, is an element of $\mathcal{R}$.

Consequence: Such a category is locally computable in the sense that

- the sets of morphisms between two objects are always finite and computable,
- finite products and fibre products are computable,
- the composite of two morphisms is always easily computable.

Reduction of geometric logic to relational languages

Definition. – *Let Σ be a first-order language.*

(i) *We denote by Σ^r the relational language derived from Σ by replacing each function symbol $f : A_1 \cdots A_n \longrightarrow B$ by a relation symbol $R_f \rightarrowtail A_1 \cdots A_n B$.*

(ii) *We denote by $\mathbb{T}^r_\Sigma$ the relational theory of language Σ^r defined by the axioms associated with the function symbols $f : A_1 \cdots A_n \longrightarrow B$ of Σ*

$$\top(x_1^{A_1} \cdots x_n^{A_n}) \vdash (\exists\, x^B)\, R_f(x_1^{A_1} \cdots x_n^{A_n} x^B),$$
$$R_f(x_1^{A_1} \cdots x_n^{A_n} x^B) \wedge R_f(x_1^{A_1} \cdots x_n^{A_n} y_B) \vdash x_B = y_B.$$

Definition. – *To every geometric formula φ of Σ, one recursively associates a geometric formula φ^r of Σ^r as follows:*

(i) *We set $\varphi = \varphi^r$ if the construction of φ does not involve any function symbol, in particular if φ has the form $R(x_1^{A_1} \cdots x_n^{A_n})$, $x^B = y^B$, $\top(x_1^{A_1} \cdots x_n^{A_n})$, $\perp(x_1^{A_1}, \cdots, x_n^{A_n})$.*

(ii) *If $\varphi = \bigvee_{i \in I} \varphi_i$, $\varphi = \varphi_1 \wedge \cdots \wedge \varphi_n$ or $\varphi(\vec{x}) = (\exists\, \vec{y})\, \theta(\vec{x}, \vec{y})$, we set*

$$\varphi^r = \bigvee_{i \in I} \varphi_i^r,\ \varphi = \varphi_1^r \wedge \cdots \wedge \varphi_n^r \ \text{ or }\ \varphi^r(\vec{x}) = (\exists\, \vec{y})\, \theta^r(\vec{x}, \vec{y}).$$

(iii) *If φ_1 is derived from φ by a substitution of variables $y^B = f(x_1^{A_1} \cdots x_n^{A_n})$, we set $\varphi_1^r = \varphi^r \wedge R_f(x_1^{A_1} \cdots x_n^{A_n} y^B)$.*

Presentation of toposes by locally computable categories

Recall:

Proposition. –
For every topos $\mathcal{E}$, there exist geometric first-order theories $\mathbb{T}$ that are "classified" by $\mathcal{E}$ in the sense that there exist equivalences

$$\mathcal{E} \xrightarrow{\sim} \mathcal{E}_{\mathbb{T}} \quad .$$

On the other hand:

Proposition. –
Let $\mathbb{T}$ be a geometric theory of language Σ defined by axioms

$$\varphi_i \vdash \psi_i \,, \quad i \in I \,.$$

Let Σ^r be the relational language associated with Σ,
and let $\mathbb{T}^r_\Sigma$ be the geometric theory of language Σ^r associated with Σ.
Finally, let $\mathbb{T}^r$ be the geometric theory of language Σ^r defined as the quotient theory of $\mathbb{T}^r_\Sigma$ by the additional axioms

$$\varphi_i^r \vdash \psi_i^r \,, \quad i \in I \,.$$

Then:

(i) *The geometric syntactic categories $\mathcal{C}_{\mathbb{T}}$ and $\mathcal{C}_{\mathbb{T}^r}$ are identified together with their canonical topologies $J_{\mathbb{T}}$ and $J_{\mathbb{T}^r}$.*

(ii) *The classifying toposes $\mathcal{E}_{\mathbb{T}}$ and $\mathcal{E}_{\mathbb{T}^r}$ of $\mathbb{T}$ and $\mathbb{T}^r$ are identified.*

(iii) *We have $\mathcal{E}_{\mathbb{T}} \cong \widehat{(\mathcal{C}^c_{\Sigma^r})}_{J_{\mathbb{T}}}$ for a topology $J_{\mathbb{T}}$ on the locally computable category $\mathcal{C}^c_{\Sigma^r}$ generated by covering presieves constructed from the axioms of $\mathbb{T}^r$.*

Simplified writing of geometric formulas

Commutation rules of first-order logic. –
Let Σ be any first-order language.

(i) *For all geometric formulas of Σ in a finite sequence of variables $\vec{x}$,*
$$\varphi(\vec{x}) \quad \text{and} \quad \varphi_i(\vec{x})\,, \quad i \in I\,,$$
the formula $\varphi(\vec{x}) \wedge \bigvee_{i\in I} \varphi_i(\vec{x})$ *is equivalent to the formula* $\bigvee_{i\in I} (\varphi(\vec{x}) \wedge \varphi_i(\vec{x}))$.

(ii) *For all geometric formulas $\varphi(\vec{x})$ and $\psi(\vec{x},\vec{y})$ of Σ,*
the formula $\varphi(\vec{x}) \wedge (\exists\, \vec{y})\, \psi(\vec{x},\vec{y})$ *is equivalent to the formula* $(\exists\, \vec{y})(\varphi(\vec{x}) \wedge \psi(\vec{x},\vec{y}))$.

(iii) *For all geometric formulas $\varphi_i(\vec{x},\vec{y})$, $i \in I$, of Σ,*
the formula $(\exists\, \vec{y}) \bigvee_{i\in I} \varphi_i(\vec{x},\vec{y})$ *is equivalent to the formula* $\bigvee_{i\in I} (\exists\, \vec{y})\varphi_i(\vec{x},\vec{y})$.

(iv) *The operations of substitution of a variable x^B into a function $x^B = f(x_1^{B_1}, \cdots, x_m^{B_m})$ commute with arbitrary unions $\bigvee$, finite intersections $\wedge$ and the operators of existential quantification over other variables $(\exists\, \vec{y})$.*

Corollary. – *Every geometric formula in a language Σ can be rewritten in the form*
$$\varphi(\vec{x}) = \bigvee_{i\in I} (\exists\, \vec{x}_i)(\varphi_{i,1}(\vec{x},\vec{x}_i) \wedge \cdots \wedge \varphi_{i,n_i}(\vec{x},\vec{x}_i))$$
where each $\varphi_{i,j}$, $i \in I$, $1 \le j \le n_i$, is an "atomic formula".

Reminder: A formula is "atomic" if it has the form $R(x_1^{A_1}, \cdots, x_n^{A_n})$ or $x^B = y^B$ or is derived from it by a finite number of substitutions $x^B = f(x_1^{B_1}, \cdots, x_m^{B_m})$.

Families of presieves associated with geometric sequents

Observation. –
Let Σ be a geometric language, and let $\mathcal{C}^c_\Sigma$ be its cartesian syntactic category.
Then:

(i) *Every atomic formula of Σ*
or more generally every finite conjunction of atomic formulas

$$\varphi(\vec{x})$$

defines an object of the category $\mathcal{C}^c_\Sigma$.

(ii) *If $\varphi(\vec{x},\vec{y})$ and $\psi(\vec{x})$ are two finite conjunctions of atomic formulas,*
forgetting the variables of the finite sequence $\vec{y}$ defines a morphism of $\mathcal{C}^c_\Sigma$

$$\varphi(\vec{x},\vec{y}) \wedge \psi(\vec{x}) \longrightarrow \psi(\vec{x}) \,.$$

Definition. – *In the context of a geometric language Σ,*
let us associate to every pair of geometric formulas in a finite sequence of variables $\vec{x}$

$$\varphi(\vec{x}) \qquad \psi(\vec{x})$$

written in simplified form

$$\begin{aligned} \varphi(\vec{x}) &= \bigvee_{i\in I} (\exists\, \vec{x}_i)\, \varphi_i(\vec{x},\vec{x}_i) \\ \psi(\vec{x}) &= \bigvee_{k\in K} (\exists\, \vec{y}_k)\, \psi_k(\vec{x},\vec{y}_k) \end{aligned}$$

(where the formulas φ_i and ψ_k are finite conjunctions of atomic formulas),
the family $\mathcal{P}_{\varphi,\psi}$ of presieves indexed by the elements $i \in I$

$$\big(\varphi_i(\vec{x},\vec{x}_i) \wedge \psi_k(\vec{x},\vec{y}_k) \longrightarrow \varphi_i(\vec{x},\vec{x}_i)\big)_{k\in K} \,.$$

Topological expression of provability

Theorem. –
Let $\mathbb{T}$ be a geometric first-order theory of language Σ.
Let $J_{\mathbb{T}}$ be the topology on the cartesian syntactic category $\mathcal{C}_{\Sigma}^{c}$ that defines the classifying topos of $\mathbb{T}$ in the sense that

$$\mathcal{E}_{\mathbb{T}} \cong \widehat{(\mathcal{C}_{\Sigma}^{c})_{J_{\mathbb{T}}}} .$$

Then a geometric sequent of Σ

$$\varphi(\vec{x}) \vdash \psi(\vec{x})$$

is provable in the theory $\mathbb{T}$ if and only if every presieve belonging to the family $\mathcal{P}_{\varphi,\psi}$ is covering for the topology $J_{\mathbb{T}}$.

Corollary. –
If the theory $\mathbb{T}$ of language Σ is defined by the family of axioms

$$\varphi_j \vdash \psi_j \,, \quad j \in J \,,$$

then a geometric sequent of Σ

$$\varphi \vdash \psi$$

is provable in $\mathbb{T}$ if and only if every presieve belonging to the family $\mathcal{P}_{\varphi,\psi}$ is covering for the topology on $\mathcal{C}_{\Sigma}^{c}$ generated by the presieves belonging to the families $\mathcal{P}_{\varphi_j,\psi_j}$, $j \in J$.

Topological translation of proofs: the cut rule

The cut rule. –
If $\varphi(\vec{x}) \vdash \psi(\vec{x})$ *and* $\psi(\vec{x}) \vdash \chi(\vec{x})$, *then* $\varphi(\vec{x}) \vdash \chi(\vec{x})$.

Topological translation. –
Write in simplified form

$$\begin{aligned}
\varphi(\vec{x}) &= \bigvee_{i \in I} (\exists\, \vec{x}_i)\, \varphi_i(\vec{x}, \vec{x}_i)\,, \\
\psi(\vec{x}) &= \bigvee_{k \in K} (\exists\, \vec{y}_k)\, \psi_k(\vec{x}, \vec{y}_k)\,, \\
\chi(\vec{x}) &= \bigvee_{\ell \in L} (\exists\, \vec{z}_\ell)\, \chi_\ell(\vec{x}, \vec{z}_\ell)\,.
\end{aligned}$$

If for every element $i \in I$ the presieve

$$(\varphi_i(\vec{x}, \vec{x}_i) \wedge \psi_k(\vec{x}, \vec{y}_k) \longrightarrow \varphi_i(\vec{x}, \vec{x}_i))_{k \in K}$$

is covering, and if for every element $k \in K$ the presieve

$$(\psi_k(\vec{x}, \vec{y}_k) \wedge \chi_\ell(\vec{x}, \vec{z}_\ell) \longrightarrow \psi_k(\vec{x}, \vec{y}_k))_{\ell \in L}$$

is covering, then for every element $i \in I$, the presieve

$$(\varphi_i(\vec{x}, \vec{x}_i) \wedge \chi_\ell(\vec{x}, \vec{z}_\ell) \longrightarrow \varphi_i(\vec{x}, \vec{x}_i))_{i \in I}$$

is covering.

Remark: One uses the three axioms of:

(1) composition,
(2) factorisation,
(3) induction.

Topological translation of proofs: the identity rule

The identity rule. –
For every variable x^A taking values in a sort A, we have

$$\top(x^A) \vdash x^A = x^A \,.$$

Topological translation. –
The monomorphism of $\mathcal{C}_\Sigma^c$

$$(x^A = x^A) \hookrightarrow \top(x^A)$$

is an isomorphism, hence is covering.

Remark:

One uses the two axioms of:

(0) identity,

(2) factorisation.

Topological translation of proofs: the equality rules

The equality rules. –

(i) *One always has* $x^A = y^A \vdash y^A = x^A$.

(ii) *One always has* $x^A = y^A \wedge y^A = z^A \vdash x^A = z^A$.

Topological translation. –

(i) *The monomorphism of* $\mathcal{C}_\Sigma^c$

$$(x^A = y^A) \wedge (y^A = x^A) \hookrightarrow (x^A = y^A)$$

is an isomorphism, hence is covering.

(ii) *The monomorphism of* $\mathcal{C}_\Sigma^c$

$$(x^A = y^A) \wedge (y^A = z^A) \wedge (x^A = z^A) \hookrightarrow (x^A = y^A) \wedge (y^A = z^A)$$

is an isomorphism, hence is covering.

Remark:

One again uses the axioms of:

(0) identity,

(2) factorisation.

Topological translation of proofs: the function substitution rule

The function substitution rule. –

If $\varphi(\vec{x})$ and $\psi(\vec{x})$ are two geometric formulas in a finite sequence of variables $\vec{x}$,
and $\varphi'(\vec{x'})$ and $\psi'(\vec{x}')$ are the two formulas derived from them
by substituting for one of the variables x^A of $\vec{x}$ a function $f(x_1'^{A_1'}, \cdots, x_m'^{A_m'}) = x^A$,
then the implication $\varphi(\vec{x}) \vdash \psi(\vec{x})$
entails the implication $\varphi'(\vec{x'}) \vdash \psi'(\vec{x'})$.

Topological translation. – Write the formulas $\varphi(\vec{x})$ and $\psi(\vec{x})$ in simplified form

$$\varphi(\vec{x}) = \bigvee_{i \in I} (\exists\, \vec{x}_i)\, \varphi_i(\vec{x}, \vec{x}_i)\,,$$
$$\psi(\vec{x}) = \bigvee_{k \in K} (\exists\, \vec{y}_k)\, \psi_k(\vec{x}, \vec{y}_k)$$

where the $\varphi_i(\vec{x}, \vec{x}_i)$ and $\psi_k(\vec{x}, \vec{y}_k)$ are finite conjunctions of atomic formulas.

The substitution $x^A = f(x_1'^{A_1'}, \cdots, x_m'^{A_m'})$ transforms these into formulas

$$\varphi_i'(\vec{x}', \vec{x}_i) \qquad \text{and} \qquad \psi_k'(\vec{x}', \vec{y}_k)\,.$$

If the presieves

$$(\varphi_i(\vec{x}, \vec{x}_i) \wedge \psi_k(\vec{x}, \vec{y}_k) \longrightarrow \varphi_i(\vec{x}, \vec{x}_i))_{k \in K}$$

are covering, so are the presieves

$$\big(\varphi_i'(\vec{x}', \vec{x}_i) \wedge \psi_k'(\vec{x}', \vec{y}_k) \longrightarrow \varphi_i'(\vec{x}', \vec{x}_i)\big)_{k \in K}$$

induced by the morphisms $\varphi_i'(\vec{x}', \vec{x}_i) \xrightarrow{x^A = f(x_1'^{A_1'}, \cdots, x_m'^{A_m'})} \varphi_i(\vec{x}, \vec{x}_i)$.

Remark: One uses axiom (3) of induction.

Topological translation of proofs: the rule of substitution of an equality in a term

Definition. – *A "term" $h(\vec{x})$ in a finite sequence of variables $\vec{x}$ taking values in a sort B can be*

$$\begin{cases} \bullet & \text{a variable of } \vec{x} \text{ associated with the sort } B, \\ \bullet & \text{a function in some of the variables of } \vec{x} \text{ taking values in the sort } B, \\ \bullet & \text{an expression obtained from a function taking values in the sort } B \\ & \text{by a finite sequence of substitutions of variables by functions.} \end{cases}$$

The rule of substitution of an equality in a term. –

Consider two terms $g(x_1'^{A_1'}, \cdots, x_m'^{A_m'})$ and $h(x_1'^{A_1'}, \cdots, x_m'^{A_m'})$ taking values in a sort A.
Consider a finite sequence of variables $\vec{x}$ containing a variable x^A associated with A,
and the sequence $\vec{x}'$ derived from $\vec{x}$ by replacing x^A by $x_1'^{A_1'}, \cdots, x_m'^{A_m'}$.
Then, for every term $k(\vec{x})$ and the terms $k_1(\vec{x}')$, $k_2(\vec{x}')$ derived from it
by the substitutions $x^A = g(x_1'^{A_1'}, \cdots, x_m'^{A_m'})$, $x^A = h(x_1'^{A_1'}, \cdots, x_m'^{A_m'})$,
the implication $\quad \top \vdash g(x_1'^{A_1'}, \cdots, x_m'^{A_m'}) = h(x_1'^{A_1'}, \cdots, x_m'^{A_m'})$
entails the implication $\quad \top \vdash k_1(\vec{x}') = k_2(\vec{x}')$.

Topological translation. – The terms g, h and k define morphisms

$$\top(\vec{x}') \overset{g}{\underset{h}{\rightrightarrows}} \top(\vec{x}) \xrightarrow{k} \top(y^B) \,.$$

The monomorphism $\quad (k \circ g = k \circ h) \hookrightarrow \top(\vec{x}')$ is covering
if the monomorphism $\quad (g = h) \hookrightarrow \top(\vec{x}')$ is covering
since there is a factorisation $\quad (g = h) \hookrightarrow (k \circ g = k \circ h)$.

The rule of substitution of an equality in a formula. –

Let two terms $g(x_1'^{A_1'}, \cdots, x_m'^{A_m'})$ and $h(x_1'^{A_1'}, \cdots, x_m'^{A_m'})$ take values in a sort A.
Let a finite sequence of variables $\vec{x}$ contain a variable x^A associated with A
and let $\vec{x}\,'$ be the sequence derived from $\vec{x}$ by replacing x^A by $x_1'^{A_1'}, \cdots, x_m'^{A_m'}$.
Then, for every geometric formula $\varphi(\vec{x})$ and the formulas $\varphi^g(\vec{x}\,')$, $\varphi^h(\vec{x}\,')$
derived from it by the substitutions $x^A = g(x_1'^{A_1'}, \cdots, x_m'^{A_m'})$, $x^A = h(x_1'^{A_1'}, \cdots, x_m'^{A_m'})$,
the implication $\top \vdash g(x_1'^{A_1'}, \cdots, x_m'^{A_m'}) = h(x_1'^{A_1'}, \cdots, x_m'^{A_m'})$
entails the equivalence $\varphi^g(\vec{x}\,') \dashv\vdash \varphi^h(\vec{x}\,')$ **.**

Topological translation. – Writing in simplified form $\varphi(\vec{x}) = \bigvee\limits_{i\in I}(\exists\,\vec{x}_i)\ \varphi_i(\vec{x},\vec{x}_i)$, we have

$$\varphi^g(\vec{x}\,') = \bigvee_{i\in I}(\exists\,\vec{x}_i)\ \varphi_i^g(\vec{x}\,',\vec{x}_i) \quad \text{and} \quad \varphi^h(\vec{x}\,') = \bigvee_{i\in I}(\exists\,\vec{x}_i)\ \varphi_i^h(\vec{x}\,',\vec{x}_i)\,.$$

The formulas $\varphi_i^g(\vec{x}\,',\vec{x}_i)$ and $\varphi_i^h(\vec{x}\,',\vec{x}_i)$ fit into cartesian squares:

$$\begin{tikzcd}
\varphi_i^g(\vec{x}\,',\vec{x}_i) \arrow[r, hook] \arrow[d] & \top(\vec{x}\,',\vec{x}_i) \arrow[d, "g"] \\
\varphi_i(\vec{x},\vec{x}_i) \arrow[r, hook] & \top(\vec{x},\vec{x}_i)
\end{tikzcd}
\qquad
\begin{tikzcd}
\varphi_i^h(\vec{x}\,',\vec{x}_i) \arrow[r, hook] \arrow[d] & \top(\vec{x}\,',\vec{x}_i) \arrow[d, "h"] \\
\varphi_i(\vec{x},\vec{x}_i) \arrow[r, hook] & \top(\vec{x},\vec{x}_i)
\end{tikzcd}$$

They contain the common subobject $\varphi_i^g(\vec{x}\,',\vec{x}_i) \wedge (g=h) = \varphi_i^h(\vec{x}\,',\vec{x}_i) \wedge (g=h)$
and the two monomorphisms

$$\varphi_i^g(\vec{x}\,',\vec{x}_i) \wedge (g=h) \hookrightarrow \varphi_i^g(\vec{x}\,',\vec{x}_i) \qquad \varphi_i^h(\vec{x},\vec{x}_i) \wedge (g=h) \hookrightarrow \varphi_i^h(\vec{x},\vec{x}_i)$$

are covering since the monomorphism $(g=h) \hookrightarrow \top(\vec{x}\,')$ is covering by hypothesis.

This rule reduces to:

The binary conjunction rule. –
Given three geometric formulas $\varphi(\vec{x}), \psi(\vec{x}), \chi(\vec{x})$
in the same finite sequence of variables $\vec{x}$*, one has the implication*

$$\varphi \vdash \psi \wedge \chi$$

if and only if one has the two implications

$$\varphi \vdash \psi \quad \textit{and} \quad \varphi \vdash \chi \,.$$

Topological translation. –
Writing in simplified form

$$\varphi = \bigvee_{i \in I} (\exists \, \vec{x}_i) \, \varphi_i(\vec{x}, \vec{x}_i) \,,$$
$$\psi = \bigvee_{k \in K} (\exists \, \vec{y}_k) \, \psi_k(\vec{x}, \vec{y}_k)$$

and

$$\chi = \bigvee_{\ell \in L} (\exists \, \vec{z}_\ell) \, \chi_\ell(\vec{x}, \vec{z}_\ell) \,,$$

one has $\quad \varphi \vdash \psi \quad$ and $\quad \varphi \vdash \chi$
if and only if, for every $i \in I$, the two families of morphisms

$$\big(\varphi_i(\vec{x}, \vec{x}_i) \wedge \psi_k(\vec{x}, \vec{y}_k) \longrightarrow \varphi_i(\vec{x}, \vec{x}_i) \big)_{k \in K}$$

and

$$\big(\varphi_i(\vec{x}, \vec{x}_i) \wedge \chi_\ell(\vec{x}, \vec{z}_\ell) \longrightarrow \varphi_i(\vec{x}, \vec{x}_i) \big)_{\ell \in L}$$

are covering.

The binary conjunction rule: cartesian squares of coverings

For all indices $i \in I$, $k \in K$, $\ell \in L$, one has a cartesian square:

$$\begin{tikzcd}
\varphi_i(\vec{x},\vec{x}_i) \wedge \psi_k(\vec{x},\vec{y}_k) \wedge \chi_\ell(\vec{x},\vec{z}_\ell) \arrow[r] \arrow[d] & \varphi_i(\vec{x},\vec{x}_i) \wedge \chi_\ell(\vec{x},\vec{z}_\ell) \arrow[d] \\
\varphi_i(\vec{x},\vec{x}_i) \wedge \psi_k(\vec{x},\vec{y}_k) \arrow[r] & \varphi_i(\vec{x},\vec{x}_i)
\end{tikzcd}$$

Since the families of morphisms

$$(\varphi_i(\vec{x},\vec{x}_i) \wedge \psi_k(\vec{x},\vec{y}_k) \longrightarrow \varphi_i(\vec{x},\vec{x}_i))_{k\in K}\ ,$$
$$(\varphi_i(\vec{x},\vec{x}_i) \wedge \chi_\ell(\vec{x},\vec{z}_\ell) \longrightarrow \varphi_i(\vec{x},\vec{x}_i))_{\ell\in L}$$

are covering, one deduces from the topological axioms

$$\begin{cases} (1) & \text{composition,} \\ (3) & \text{induction,} \end{cases}$$

that, for every $i \in I$, the family of morphisms

$$(\varphi_i(\vec{x},\vec{x}_i) \wedge \psi_k(\vec{x},\vec{y}_k) \wedge \chi_\ell(\vec{x},\vec{z}_\ell) \longrightarrow \varphi_i(\vec{x},\vec{x}_i))_{k\in K,\ell\in L}$$

is covering.

This translates the implication $\quad \varphi \vdash \psi \wedge \chi$ since

$\psi \wedge \chi$ is written in simplified form

$$\psi \wedge \chi = \bigvee_{\substack{k\in K \\ \ell\in L}} (\exists\, \vec{y}_k)(\exists\, \vec{z}_\ell)\,(\psi_k(\vec{x},\vec{y}_k) \wedge \chi_\ell(\vec{x},\vec{z}_\ell)).$$

Topological translation of proofs: the disjunction rule

The disjunction rule. –
Given geometric formulas $\varphi(\vec{x})$ *and* $\varphi_k(\vec{x})$, $k \in K$,
one has the implications $\varphi_k(\vec{x}) \vdash \varphi(\vec{x})$ *for every* $k \in K$
if and only if one has the implication $\bigvee_{k \in K} \varphi_k(\vec{x}) \vdash \varphi(\vec{x})$.

Topological translation.– Write in simplified forms

$$\varphi(\vec{x}) = \bigvee_{i \in I} (\exists\, \vec{x}_i)\, \varphi_i(\vec{x}, \vec{x}_i)\,,$$

$$\varphi_k(\vec{x}) = \bigvee_{\ell \in L_k} (\exists\, \vec{y}_{k,\ell})\, \varphi_{k,\ell}(\vec{x}, \vec{y}_{k,\ell})$$

and therefore

$$\bigvee_{k \in K} \varphi_k(\vec{x}) = \bigvee_{k \in K} \; \bigvee_{\ell \in L_k} (\exists\, \vec{y}_{k,\ell})\, \varphi_{k,\ell}(\vec{x}, \vec{y}_{k,\ell}).$$

The implication $\bigvee_{k \in K} \varphi_k(\vec{x}) \vdash \varphi(\vec{x})$
translates to the fact that, for all indices $k \in K$, $\ell \in L_k$, the family

$$\big(\varphi_i(\vec{x}, \vec{x}_i) \wedge \varphi_{k,\ell}(\vec{x}, \vec{y}_{k,\ell}) \longrightarrow \varphi_{k,\ell}(\vec{x}, \vec{y}_{k,\ell})\big)_{i \in I}$$

is covering.
This is also the translation of the family of implications

$$\varphi_k(\vec{x}) \vdash \varphi(\vec{x})\,, \quad k \in K\,.$$

Remark:
Thus, the disjunction rule is a tautology in its topological translation.

The existential quantification rule. –
Given geometric formulas $\psi(\vec{x})$ *and* $\varphi(\vec{x},\vec{y})$,
the implication $\varphi(\vec{x},\vec{y}) \vdash_{\vec{x},\vec{y}} \psi(\vec{x})$
is equivalent to the implication $(\exists\,\vec{y})\varphi(\vec{x},\vec{y}) \vdash_{\vec{x}} \psi(\vec{x})$.

Topological translation.– Write in simplified forms

$$\psi(\vec{x}) = \bigvee_{k\in K} (\exists\,\vec{x}_k)\,\psi_k(\vec{x},\vec{x}_k)\,,$$
$$\varphi(\vec{x},\vec{y}) = \bigvee_{\ell\in L} (\exists\,\vec{z}_\ell)\,\varphi_\ell(\vec{x},\vec{y},\vec{z}_\ell)\,,$$

and therefore

$$(\exists\,\vec{y})\,\varphi(\vec{x},\vec{y}) = \bigvee_{\ell\in L} (\exists\,\vec{y})(\exists\,\vec{z}_\ell)\,\varphi_\ell(\vec{x},\vec{y},\vec{z}_\ell).$$

The implication $\varphi(\vec{x},\vec{y}) \vdash_{\vec{x},\vec{y}} \psi(\vec{x})$
translates to the fact that, for every index $\ell \in L$, the family

$$(\varphi_\ell(\vec{x},\vec{y},\vec{z}_\ell) \wedge \psi_k(\vec{x},\vec{x}_k) \longrightarrow \varphi_\ell(\vec{x},\vec{y},\vec{z}_\ell))_{k\in K}$$

is covering.
This is also the topological translation of the implication

$$(\exists\,\vec{y})\,\varphi(\vec{x},\vec{y}) \vdash_{\vec{x}} \psi(\vec{x})\,.$$

Remark:
Thus, the existential quantification rule is also a tautology
in its topological translation.

Topological translation of proofs: the commutation rules

The commutation rules. –

(i) *(Distributivity) For all geometric formulas* $\varphi(\vec{x})$ *and* $\varphi_i(\vec{x})$, $i \in I$,
the formula $\varphi(\vec{x}) \wedge \bigvee\limits_{i\in I} \varphi_i(\vec{x})$
is equivalent to the formula $\bigvee\limits_{i\in I} (\varphi(\vec{x}) \wedge \varphi_i(\vec{x}))$.

(ii) *(Frobenius rule) For all geometric formulas* $\varphi(\vec{x})$ *and* $\psi(\vec{x},\vec{y})$,
the formula $\varphi(\vec{x}) \wedge (\exists\,\vec{y})\,\psi(\vec{x},\vec{y})$
is equivalent to the formula $(\exists\,\vec{y})(\varphi(\vec{x}) \wedge \psi(\vec{x},\vec{y}))$.

(iii) *For all geometric formulas* $\varphi_i(\vec{x},\vec{y})$, $i \in I$,
the formula $(\exists\,\vec{y})\big(\bigvee\limits_{i\in I} \varphi_i(\vec{x},\vec{y})\big)$
is equivalent to the formula $\bigvee\limits_{i\in I} (\exists\,\vec{y})\varphi_i(\vec{x},\vec{y})$.

Topological translation.–

In all three cases (i), (ii) and (iii), the two formulas considered have the same rewriting in simplified form.

Indeed, the rewriting of geometric formulas in simplified form rests on these three commutation rules.

In other words, these commutation rules are contained in the topological translation process of formulas.